Solution Set Splitting at Low Energy Levels in Schrödinger Equations with Periodic and Symmetric Potential

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Dedicated to Professor Norman Dancer on the occasion of his 60th birthday

Abstract

The time-independent superlinear Schrödinger equation with spatially periodic and positive potential admits sign-changing two-bump solutions if the set of positive solutions at the minimal nontrivial energy level is the disjoint union of period translates of a compact set. Assuming a reflection symmetric potential we give a condition on the equation that ensures this splitting property for the solution set. Moreover, we provide a recipe to explicitly verify the condition, and we carry out the calculation in dimension one for a specific class of potentials.

1. Introduction and Statement of Results

Solutions of the stationary nonlinear Schrödinger equation

$$-\Delta u + V(x)u = |u|^{p-2}u, \qquad u \in H^1(\mathbb{R}^N), \tag{1.1}$$

yield standing waves of the associated time-dependent nonlinear Schrödinger equation. We are interested in the case where V is positive and periodic.

Starting with a paper by Coti Zelati and Rabinowitz [9] there has been a lot of activity regarding the existence of so-called "multibump" solutions of (1.1), see the survey by Rabinowitz [20] and the references in [1]. Roughly, one assumes the existence of an isolated mountain pass solution u_0 and obtains solutions near the sum of multiple translated copies

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of u_0 and $-u_0$. Kabeya and Tanaka [15] gave the first (parameter-dependent) example of potentials V such that the assumption of existence of an isolated u_0 is satisfied.

Taking a somewhat different approach, in [3] we constructed sign-changing *two-bump* solutions under the weaker assumption that the solution set at the minimal energy level c_0 splits into translates of a compact set, see condition $(\mathrm{S})_{c_0}$ below. We also gave parameter-dependent examples where this condition is satisfied, covering wider classes of potentials than considered in [15].

Initially, multibump solutions appeared as homoclinics in Hamiltonian systems in the work of Séré [23, 24] and Coti Zelati and Rabinowitz [8]. Only countability of the number of homoclinic orbits needed to be assumed. In that setting there also exist results that carry out the multibump construction without excluding the appearance of continua of homoclinics, see [18, 21, 26]. Moreover, there are many results about the existence of multibump solutions in Hamiltonian systems with slowly oscillating forcing term; for this type of result we mention the papers [4–7, 10, 22]. This shows that for Hamiltonian systems the known results about multibump solutions are considerably better.

Our aim in the present paper is to provide more examples of potentials in (1.1) where the splitting condition $(\mathrm{S})_c$ holds, focusing on concrete, calculable examples. It turns out that generally slowly oscillating forcing terms induce this property, reminiscent of the results for Hamiltonian systems. The advantage of our results lies in the computability. In dimension 1 we carry out the computations and show that our method leads to reasonable results.

There is one drawback in that [3] only constructs *two-bump* solutions. We hope to remedy this situation in a forthcoming paper, by constructing multibump solutions only assuming the splitting condition.

Set $2^* := 2N/(N-2)$ if $N \geq 3$ and $2^* := \infty$ if $N = 1, 2$, and let $p \in (2, 2^*)$. Denote the ith coordinate of $x \in \mathbb{R}^N$ by x^i and set $\partial_i := \partial/\partial x^i$. For the statement of our results assume the following hypotheses on V:

(V1) $V \in C^1(\mathbb{R}^N)$ and V' is Lipschitz continuous.

(V2) $\inf V(\mathbb{R}^N) > 0$.

(V3) V is periodic in every coordinate x^i, with minimal period $\tau_i > 0$ in the ith coordinate.

(V4) V is even in x^i, for all $i = 1, 2, \ldots, N$.

The symmetry condition (V4) above has been considered by other authors, see for example [11, 14, 27].

The continuously differentiable functional

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p\,\mathrm{d}x,$$

defined on the space $E := H^1(\mathbb{R}^N)$ with norm given by $\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,\mathrm{d}x$, defines a variational setting for solving (1.1): Weak solutions of (1.1) correspond to critical points of J. Denote

$$\begin{aligned} K &:= \{ u \in E\backslash\{0\} \mid J'(u) = 0 \}, \\ K_+ &:= \{ u \in K \mid u > 0 \}, \\ K_- &:= \{ u \in K \mid u < 0 \}, \\ K^c &:= \{ u \in K \mid J(u) \le c \}, \end{aligned}$$

and

$$K_\pm^c := K^c \cap K_\pm$$

for $c \in \mathbb{R}$. The existence of a nontrivial solution of (1.1), and hence $K \neq \varnothing$, was first shown by Rabinowitz, cf. [19]. The least nontrivial energy level

$$c_0 := \inf J(K) \tag{1.2}$$

exists, is positive, and is achieved by a positive function. Moreover, c_0 is the least mountain pass value. These facts are well known; for proofs see for example [3].

Define $T \in \mathcal{L}(\mathbb{R}^N)$ to be the diagonal matrix with diagonal elements $\tau_1, \tau_2, \dots, \tau_N$. The set $\mathbb{Z}^N$ induces an action "$\star$" on E by $(d \star u)(x) := u(x - Td)$ for $u \in E$, $d \in \mathbb{Z}^N$ and $x \in \mathbb{R}^N$ (translation in steps of period length). It follows that J is invariant under this action since V is "T-periodic in x".

For $c < 2c_0$ we say that K_+ *splits at the level* c if

(S)$_c$
There is a compact subset $\mathcal{K} \subseteq K_+^c$ such that the following hold:

(i) $K_+^c = \mathbb{Z}^N \star \mathcal{K}$,

(ii) $\mathcal{K} \cap (\mathbb{Z}^N\backslash\{0\}) \star \mathcal{K} = \varnothing$.

By condition (V1) $V \in W^{2,\infty}(\mathbb{R}^N)$, the Sobolev space of functions in $L^\infty(\mathbb{R}^N)$ with weak first and second partial derivatives in $L^\infty(\mathbb{R}^N)$. We introduce an integral condition for the problem (1.1):

(I)$_c$
If $u \in K_+^c$ is even in x^i for some $i \in \{ 1, 2, \dots, N \}$, then

$$\int_{\mathbb{R}^N} u^2 \partial_i^2 V \,\mathrm{d}x \le 0.$$

We also say that a potential V with (V1)–(V4) satisfies (I)$_c$ if (I)$_c$ holds for the corresponding Eq. (1.1). Our main result reads:

1.1 Theorem. *Suppose that V satisfies* (V1)–(V4) *and that $c \in [c_0, 2c_0)$. Then* (I)$_c$ *implies* (S)$_c$.

The previous theorem utilizes the reflection symmetry of V at planes $\{ x^i = 0 \}$ with arguments in the spirit of the *moving plane method* [12]. There one fixes a positive solution u and considers certain extrema of continua of hyperplanes X such that u and its reflection at X are ordered on one side of X. In our work here we consider a *discrete* set of hyperplanes parallel to the coordinate axes, locked with $x^i = k\tau_i$, $k \in \mathbb{Z}$, and apply reflections to solutions from K_+^c. *This* set may include a continuum. In that sense our use of this technique is inverse to the moving plane method, and one may speak of hyperplanes *skipping at period intervals*.

The following theorem helps to check the validity of $(\mathrm{I})_c$ for a given potential V and $c \in [c_0, 2c_0)$. We state it here since it may be of independent interest. Note that it is proved in much more generality in Sect. 3 below.

1.2 Theorem. *Suppose that V satisfies* (V1), (V2), *and* $\|V\|_{C^1} < \infty$. *Fix $\varepsilon > 0$. Then there are positive constants C_1, C_2, C_3 and C_4 that depend only on ε, p,* $\inf V$, *and on an upper bound for $\|V\|_{C^1}$, and that can be estimated explicitly, with the following property: Given any $u \in K_+^{2c_0-\varepsilon}$ denote by $\mathcal{M}$ the set of local maximum points of u, and denote by x_0 the center of mass of* $\operatorname{conv}(\mathcal{M})$. *Then*

$$C_3 \mathrm{e}^{-C_1|x-x_0|} \le u(x)^2 \le C_4 \mathrm{e}^{-C_2|x-x_0|}$$

for all $x \in \mathbb{R}^N$.

This theorem leads to the construction of slowly oscillating potentials V that satisfy $(\mathrm{I})_c$, as follows:

1.3 Theorem. *Suppose that W satisfies* (V1)–(V4) *in place of V, and that it is* 1*-periodic in all coordinates. Also assume for $i = 1, 2, 3, \ldots, N$ that $\partial_i^2 W$ exists in the classical sense in a neighborhood of $\{x^i = 0\}$ and that it is continuous and negative in that neighborhood. If $T \in \mathcal{L}(\mathbb{R}^N)$ is a diagonal matrix with positive diagonal elements $\tau_1, \tau_2, \ldots, \tau_N$, define $V_T(x) := W(T^{-1}x)$ for $x \in \mathbb{R}^N$. Conditions* (V1)–(V4) *remain valid for V_T in place of V, now with the periods τ_i. Then, given $c \in [c_0, 2c_0)$, there is a diagonal matrix with positive diagonal elements $T_0 \in \mathcal{L}(\mathbb{R}^N)$, only depending on c, p and the data of W in a way that can be made explicit, such that $V := V_T$ satisfies* $(\mathrm{I})_c$ *for $T \ge T_0$.*

1.4 Remark. A potential W as in the preceding theorem can be constructed easily: Suppose that $\varphi \in C^1(\mathbb{R})$ is positive, even, and 1-periodic. Also suppose that φ' is Lipschitz continuous, that φ'' exists classically near 0, and that $\varphi''(0) < 0$. Then $W(x) := \prod_{i=1}^N \varphi(x^i)$ satisfies all requirements of the theorem.

1.5 Example. We demonstrate that Theorems 1.1 and 1.3 yield reasonable concrete examples for functions V that satisfy $(\mathrm{S})_{c_0}$, at least in dimension one. We specialize to the case $p = 20$ and consider the equation

$$-u'' + Vu = |u|^{18}u, \qquad u \in H^1(\mathbb{R}^N) \tag{1.3}$$

with V given in Fig. 1. Then $(\mathrm{S})_{c_0}$ holds for (1.3).

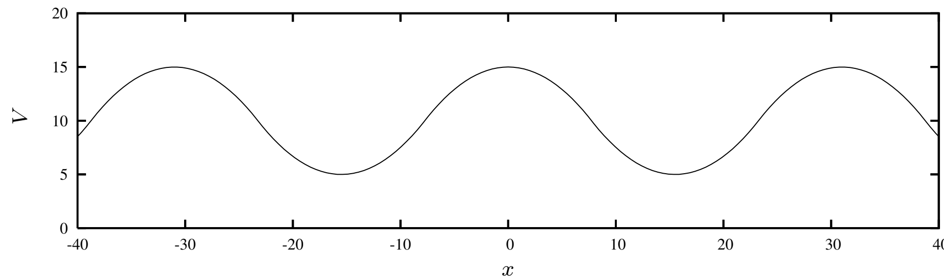

Figure 1: V with $\min V = 5$, $\max V = 15$, and period 31.

The paper is structured as follows: In Section 2 we prove Theorem 1.1. In Section 3 we prove Theorem 1.2 in a more general setting. This result is independent of Section 2. At the end of Section 3 one finds the proof of Theorem 1.3. The recipe for the calculations of Example 1.5 is explained in Section 4. Throughout we denote by $B_R(x) \subseteq \mathbb{R}^N$ the closed ball with center x and radius R.

2. Periodicity and Symmetry

We prove Theorem 1.1 in a more general setting, replacing the nonlinear term in (1.1) by a function f and considering

$$-\Delta u + V(x)u = f(u), \qquad u \in H^1(\mathbb{R}^N) \tag{2.1}$$

instead. We have refrained from considering an x-dependency in the nonlinearity, even though this could probably be done. In that case one would have to account for interactions between f and V. To keep things simple, using

$$F(u) := \int_0^u f(s)\,\mathrm{d}s$$

we assume (V1)–(V4) and the following:

(F1) $f \in C^1(\mathbb{R})$, and f' is Hölder continuous on bounded subsets of $\mathbb{R}$.

(F2) $f(u) = o(|u|)$ as $u \to 0$.

(F3) $|f'(u)| \le a(1 + |u|^{p-2})$ for $u \in \mathbb{R}$, with some $p \in (2, 2^*)$.

(F4) $f'(u)u^2 \ge (\theta - 1)f(u)u > 0$ for $u \ne 0$, with some $\theta > 2$.

2.1 Remark. Conditions (V1) and (F1) imply that every solution of (2.1) is in $C^{3,\alpha}$ for some $\alpha > 0$. We do not strive for the most general regularity assumptions here.

Using the energy functional

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x$$

we reuse the definitions of sets of critical points of J given in Section 1. Since here we do not assume oddness of f we use

$$c_0 := \inf J(K_+) \tag{2.2}$$

instead of (1.2).

2.2 Lemma. *Suppose we are given $i \in \{1, 2, \dots, N\}$ and $u \in K_+$ that is even in x^i, and such that*

$$\int_{\mathbb{R}^N} u^2 \partial_i^2 V\,\mathrm{d}x \le 0. \tag{2.3}$$

If $v \in E$ is odd in x^i, if $v(x) \neq 0$ for $x^i \neq 0$, and if

$$-\Delta v + V(x)v = \mu f'(u)v \tag{2.4}$$

for some $\mu \in \mathbb{R}$, then $\mu < 1$.

Proof. The idea of the proof is roughly the following: If v is as in the statement and solves (2.4) with $\mu \ge 1$ then v oscillates at least as fast as $\partial_i u$ by (2.3) and by differentiating (2.1) with respect to x^i. We also see that $v \not\equiv \partial_i u$ because $\partial_i V \not\equiv 0$. It therefore follows from $\partial_i u = v = 0$ on $\{x^i = 0\}$ and $\partial_i u \to 0$ as $|x| \to \infty$ that v has a zero in $\{x^i > 0\}$. Contradiction!

Set $\Omega := \{x \in \mathbb{R}^N \mid x^i > 0\}$ and define smooth functionals $\Phi, \Psi \colon H_0^1(\Omega) \to \mathbb{R}$ by setting

$$\Phi(v) := \int_\Omega (|\nabla v|^2 + Vv^2)\,\mathrm{d}x \qquad \text{and} \qquad \Psi(v) := \int_\Omega f'(u)v^2\,\mathrm{d}x.$$

Also consider the set $\mathcal{S} := \{v \in H_0^1(\Omega) \mid \Psi(v) = 1\}$. Then $\mathcal{S}$ is a smooth closed submanifold of $H_0^1(\Omega)$.

The generalized eigenvalue problem

$$-\Delta v + V(x)v = \mu f'(u)v, \qquad x \in \Omega,$$

has the eigenvalue μ and the corresponding eigenvector $v \in \mathcal{S}$ if and only if v is a critical point of $\Phi|_{\mathcal{S}}$ with $\Phi(v) = \mu$.

Since u decays exponentially at infinity, and since f' is Hölder continuous at $u = 0$, $f'(u(x))$ is bounded and decays exponentially at infinity. Hence Ψ is weakly sequentially continuous, and $\mathcal{S}$ is weakly sequentially closed. Moreover, Φ is weakly sequentially lower semicontinuous. Therefore Φ attains its minimum on $\mathcal{S}$ in an element v_0 with eigenvalue μ_0. Arguing as in the proof of [25, Theorem 2.5] it follows that μ_0 is simple, and we may assume that $v_0 > 0$. The positivity of u implies that $f'(u(x)) > 0$, and two eigenfunctions v_1, v_2 with eigenvalues $\mu_1 \neq \mu_2$ satisfy

$$\int_\Omega f'(u) v_1 v_2 \,\mathrm{d}x = 0.$$

Hence all eigenfunctions except v_0 change sign.

Given v and μ as in the statement of the lemma, we may assume that $v > 0$ on Ω. It follows from the considerations above that $v = v_0$ and $\mu = \mu_0$. Note that by Remark 2.1 it holds that

$$-\Delta \partial_i u + V \partial_i u = f'(u) \partial_i u - u \partial_i V. \tag{2.5}$$

Set $w := s\partial_i u$ where $s > 0$ is chosen such that $\Psi(w) = 1$. Recall that $\partial_i V = 0$ on $\{ x^i = 0 \}$ because V is even in x^i. Then (2.5) implies

$$\mu \leq \Phi(w) = 1 - s^2 \int_\Omega u \partial_i u \partial_i V \,\mathrm{d}x = 1 + \frac{s^2}{2} \int_\Omega u^2 \partial_i^2 V \,\mathrm{d}x \leq 1 \tag{2.6}$$

since, by assumption and by the evenness of u^2 and $\partial_i^2 V$, the last integral term is nonpositive. If $\mu = 1$ were true then (2.6) would be an equality, w a minimum point of $\Phi|_{\mathcal{S}}$, and hence $\partial_i u$ would solve (2.4) with $\mu = \mu_0 = 1$. Equation (2.5) would imply, together with $u > 0$, that $\partial_i V \equiv 0$. But this would contradict (V3). Hence we have proved $\mu < 1$. □

2.1. Proof of Theorem 1.1

Suppose we are given $c \in [c_0, 2c_0)$ such that $(\mathrm{I})_c$ holds. Consider the action of $\mathbb{Z}^N$ on itself induced by addition. We will build an equivariant map $\alpha \colon K_+^c \to \mathbb{Z}^N$ such that $\alpha^{-1}(0)$ is compact. Then $(\mathrm{S})_c$ is satisfied with $\mathcal{K} := \alpha^{-1}(0)$.

First we fix $i \in \{ 1, 2, \dots, N \}$ and construct the ith component α^i of α. For $k \in \mathbb{Z}$ denote by

$$\Omega_k := \{ x \in \mathbb{R}^N \mid x^i < k\tau_i \}$$

an affine half-space, and by $\rho_k \colon \mathbb{R}^N \to \mathbb{R}^N$,

$$\rho_k(x) := (x^1, x^2, \dots, x^{i-1}, 2k\tau_i - x^i, x^{i+1}, \dots, x^N)$$

reflection at $\partial\Omega_k$. For $u \in K_+^c$ set

$$\mathcal{A}(u) := \{ k \in \mathbb{Z} \mid u > u \circ \rho_k \text{ on } \Omega_k \text{ and } \partial_i u < 0 \text{ on } \partial\Omega_k \},$$
$$\mathcal{B}(u) := \{ k \in \mathbb{Z} \mid u < u \circ \rho_k \text{ on } \Omega_k \text{ and } \partial_i u > 0 \text{ on } \partial\Omega_k \},$$

and

$$\alpha^i(u) := \inf \mathcal{A}(u).$$

Below we show that $\alpha^i(u)$ is finite for every $u \in K_+^c$ and that α^i is continuous. We obtain a continuous map $\alpha \colon K_+^c \to \mathbb{Z}^N$ with components α^i. It is obvious that α is equivariant with respect to the action of $\mathbb{Z}^N$. To show that $\alpha^{-1}(0)$ is compact, assume that we are given a sequence $(u_n) \subseteq \alpha^{-1}(0)$. Since $J(u_n) \leq c$, the standard splitting lemma, cf. [3, Proposition 2.5], yields $u_0 \in K_+^c$ and a sequence $(d_n) \subseteq \mathbb{Z}^N$ such that, after passing to a subsequence, $d_n \star u_n \to u_0$. By equivariance and continuity,

$$d_n = d_n + \alpha(u_n) = \alpha(d_n \star u_n) \to \alpha(u_0).$$

Hence $d_n = \alpha(u_0)$ for n large, and $u_n \to (-\alpha(u_0)) \star u_0$ as $n \to \infty$. Since $(-\alpha(u_0)) \star u_0 \in \alpha^{-1}(0)$ this proves compactness of $\alpha^{-1}(0)$, and we conclude.

It remains to show that α^i is continuous for fixed i. The basic idea to do this is the following: Arguments similar to those used in the moving plane method (we follow the presentation in [16]) yield that $\alpha^i \colon K_+^c \to \mathbb{Z}$ is well defined, and continuous outside of points $u \in K_+^c$ with the following property: u is even in x^i, and the kernel of $-\Delta + V - f'(u)$ contains a nonzero element that is odd in x^i and has exactly one sign change. In turn, the existence of such u is excluded by Lemma 2.2.

Let m denote the minimum of V. We introduce the notation

$$b_1 := \sup\{\, u_0 \geq 0 \mid \forall u \in (0, u_0]\colon\ f(u) \leq mu/2 \,\}.$$

If $k \in \mathbb{Z}$ and $u \in K_+$, below we will frequently consider $\bar{u} := u - u \circ \rho_k$. It holds that

$$-\Delta \bar{u} + (V - g)\bar{u} = 0 \qquad \text{in } \mathbb{R}^N, \tag{2.7}$$

where we have set

$$g(x) := \int_0^1 f'(su(x) + (1 - s)u(\rho_k(x)))\, \mathrm{d}s. \tag{2.8}$$

This follows since by (V3) and (V4) $V \circ \rho_k = V$.

First we show that

$$-\infty < \alpha^i(u) < \infty \qquad \text{for all } u \in K_+^c. \tag{2.9}$$

Pick some $u \in K_+^c$ and $0 < R_0 < R_1$ with the following properties:

$$\begin{aligned} &\max u(\mathbb{R}^N \backslash B_{R_0}(0)) \leq b_1, \\ &\max u(\mathbb{R}^N \backslash B_{R_1}(0)) < \min u(B_{R_0}(0)). \end{aligned}$$

Suppose that $k \in \mathbb{Z}$ and $k \geq R_1/\tau_i$. Set $\bar{u} := u - u \circ \rho_k$. Then $\bar{u} > 0$ in $B_{R_0}(0)$, $\bar{u} = 0$ on $\partial\Omega_k$, $\bar{u}(x) \to 0$ as $|x| \to \infty$, and $0 \leq g(x) \leq m/2$ for $x \in \Omega_k \backslash B_{R_0}(0)$. Since $\bar{u}$ satisfies (2.7), the strong maximum principle implies that $\bar{u} > 0$ in Ω_k and $\partial_i \bar{u} < 0$ on $\partial\Omega_k$. Hence $k \in \mathcal{A}(u)$. In the same way we see that $-k \in \mathcal{B}(u)$. We have thus shown that $[R_1/\tau_i, \infty) \cap \mathbb{Z} \subseteq \mathcal{A}(u)$ and $(-\infty, -R_1/\tau_i] \cap \mathbb{Z} \subseteq \mathcal{B}(u)$. As $\mathcal{A}(u) \cap \mathcal{B}(u) = \varnothing$, we obtain $\alpha^i(u) \in [-R_1/\tau_i - 1, R_1/\tau_i + 1]$, proving (2.9).

We now proceed to prove that α^i is continuous, starting with upper semicontinuity. Note that on K_+ the E- and C^1-topologies coincide. This follows for example from [2, Thm. B.2(a)], considering that the time-1-map φ^1 is bijective on K, where φ is the parabolic semiflow induced by the parabolic equation related to (1.1). That part of the theorem applies here although we are working on $\mathbb{R}^N$ instead of a bounded domain.

Fix some $u_0 \in K_+^c$. Set $k_0 := \alpha^i(u_0)$ and pick $x_0 \in \partial\Omega_{k_0}$. There is $R > 0$ such that $|u_0| \leq b_1$ on $\mathbb{R}^N \backslash B_R(x_0)$. Moreover, there is $\delta > 0$ such that $u > u \circ \rho_0$ on $B_R(x_0) \cap \Omega_{k_0}$, and $\partial_i u < 0$ on $B_R(x_0) \cap \partial\Omega_{k_0}$ if $\|u - u_0\|_{C^1} \leq \delta$. Suppose now that $u \in K_+^c$ satisfies $\|u - u_0\|_{C^1} \leq \delta$. Set $\bar{u} := u - u \circ \rho_{k_0}$. Then $\bar{u} > 0$ on $B_R(x_0) \cap \Omega_{k_0}$, $\partial_i u < 0$ on $B_R(x_0) \cap \partial\Omega_{k_0}$, $\bar{u} = 0$ on $\partial\Omega_{k_0}$, $\bar{u}(x) \to 0$ as $|x| \to \infty$, and $0 \leq g(x) \leq m/2$ for $x \in \Omega_{k_0} \backslash B_R(x_0)$. Again, (2.7) implies by the strong maximum principle that $k_0 \in \mathcal{A}(u)$ and hence $\alpha^i(u) \leq k_0 = \alpha^i(u_0)$. This proves upper semicontinuity of α^i at u_0.

The most involved part of the proof is to show lower semicontinuity of α^i. It is here that condition $(\mathrm{I})_c$ plays a fundamental role through an application of Lemma 2.2. Suppose that we are given $(u_n) \subseteq K_+^c$, with $u_n \to u_0$ as $n \to \infty$. It was shown in the proof of [3, Proposition 5.2] that there are positive constants D_1 and D_2 such that

$$\int_{\mathbb{R}^N \backslash B_r(0)} (|\nabla u_n|^2 + u_n^2)\,\mathrm{d}x \leq \|u_n\|^2 D_1 \mathrm{e}^{-D_2 r}$$

for all $n \in \mathbb{N}$. Using the boundedness of $\|u_n\|^2$ and [13, Theorem 8.17] we therefore obtain positive constants D_3 and D_4 such that

$$u_n(x) \leq D_3 \mathrm{e}^{-D_4 |x|}$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$. Note that for the following argument it is immaterial whether these constants depend on u_0 or not. We infer that there is $R_0 > 0$ such that

$$\sup_{x \in \mathbb{R}^N \backslash B_{R_0}(0)} u_n(x) \leq b_1 \qquad \text{for all } n \in \mathbb{N}.$$

Moreover, since $u_n \to u_0$ in C^1 there is $R_1 > 0$ such that

$$\sup_{x \in \mathbb{R}^N \backslash B_{R_1}(0)} u_n(x) \leq \min_{x \in B_{R_0}(0)} u_n(x).$$

As in the proof of (2.9) it follows that

$$\alpha^i(u_n) \in [-R_1/\tau_i - 1, R_1/\tau_i + 1] \qquad \text{for all } n \in \mathbb{N}.$$

After passing to a subsequence and translating suitably we may therefore assume that $\alpha^i(u_n) = 0$ for all $n \in \mathbb{N}$.

Set $\bar{u}_n := u_n - u_n \circ \rho_0$ for $n = 0, 1, 2, \ldots$. Then $\bar{u}_n > 0$ in Ω_0 and $\bar{u}_n \to \bar{u}_0$ in $C^1(\mathbb{R}^N)$. Therefore $\bar{u}_0 \geq 0$ in $\overline{\Omega_0}$, and moreover, (2.7) holds with $\bar{u}$ replaced by $\bar{u}_0$. If we can exclude that $\bar{u}_0$ vanishes identically on Ω_0 then the strong maximum principle yields that

$\alpha^i(u_0) \le 0$ and we conclude. Note that in this situation it is not necessary that g in (2.8) (with $\bar{u}$ replaced by $\bar{u}_0$) satisfies $g \le m$.

To prove lower semicontinuity it therefore remains to show that $\bar{u}_0$ does not vanish identically. Arguing by contradiction, assume that $\bar{u}_0 \equiv 0$ or, in other words, that u_0 is even in x^i. Abusing notation we identify ρ_0 with the element from $\mathcal{L}(E)$ sending u to $u \circ \rho_0$. Note that $\rho_0^{-1} = \rho^T = \rho_0$, where ρ_0^T denotes the adjoint of ρ_0. Consider the complementary orthogonal projections

$$P_V := \frac{1}{2}(I + \rho_0)$$
$$P_W := \frac{1}{2}(I - \rho_0)$$

with images V and W. Here V is the subspace of functions even in x^i and W is the subspace of functions odd in x^i.

Set $\Gamma := \nabla J$. Since J is invariant with respect to ρ_0, Γ is equivariant. Let X and Y denote kernel and range of $\Gamma'(u_0)$. Note that X is finite-dimensional. It is easily seen that $u_0 \in V$ implies that $\Gamma'(u_0)$ and ρ_0 commute. Hence X and Y are invariant for ρ_0. From this it follows that P_V, P_W, P_X and P_Y commute pairwise, where P_X and P_Y denote the orthogonal projections onto X and Y.

The implicit function theorem yields a local map $h \colon X \to Y$ at 0 such that

$$y = h(x) \quad \text{if and only if} \quad P_Y \Gamma(u_0 + x + y) = 0 \tag{2.10}$$

for $x \in X$ and $y \in Y$ near 0. Moreover, $h(0) = 0$ and $h'(0) = 0$. Similarly, we look at the restriction of J to V. The subspace V coincides with the space of fixpoints of ρ_0. Hence $\Gamma(V) \subseteq V$. Using these properties we obtain a local map $h_V \colon X \cap V \to Y \cap V$ at 0 such that (2.10) holds with h replaced by h_V if $x \in X \cap V$ and $y \in Y \cap V$ near 0. From this it follows that $h(x) = h_V(x)$ for $x \in X \cap V$ near 0 and thus

$$P_W h(x) = 0 \qquad \text{for } x \in X \cap V \text{ near } 0. \tag{2.11}$$

Set $v_n := P_X P_V (u_n - u_0)$ and $w_n := P_X P_W (u_n - u_0)$, so

$$u_n = u_0 + v_n + w_n + h(v_n + w_n) \qquad \text{for large } n.$$

Taking (2.11) into account, this and

$$h(v_n + w_n) = h(v_n) + \int_0^1 h'(v_n + s w_n) w_n \, \mathrm{d}s$$

yield $\bar{u}_n/2 = P_W u_n = w_n + o(\|w_n\|)$. Recall that $\bar{u}_n \ne 0$ since $\alpha^i(u_n) = 0$. Therefore there exists $w_0 \in W \cap X$ with $\|w_0\| = 1$ such that $\bar{u}_n / \|\bar{u}_n\| \to w_0$ as $n \to \infty$, after passing to a subsequence. Moreover, $\bar{u}_n > 0$ on Ω_0 implies that $w_0 \ge 0$. Since $w_0 \ne 0$ and w_0 satisfies

$$-\Delta w_0 + V w_0 = f'(u_0) w_0$$

we obtain $w_0 > 0$ on Ω_0. Oddness of w_0 in x^i and Lemma 2.2 yield

$$\int_{\mathbb{R}^N} u_0^2 \partial_i^2 V \,\mathrm{d}x > 0,$$

contradicting assumption $(\mathrm{I})_c$. This concludes the proof of Theorem 1.1.

3. Uniform Decay Estimates

In this section we use a different set of assumptions as in the preceding section since the results are independent. It poses no additional difficulties to prove them in a less restrictive setting. In particular we allow the nonlinearity to depend on x and therefore consider

$$-\Delta u + V(x)u = f(x,u),\ u \in H^1(\mathbb{R}^N). \tag{3.1}$$

For V we assume:

(V1′) V is Hölder continuous.

(V2′) $\inf V(\mathbb{R}^N) > 0$.

(V3′) V is bounded.

We set $F(x,u) := \int_0^u f(x,s)\,\mathrm{d}s$ and assume:

(F1′) f is differentiable in u for almost every x, and $\partial_u f$ is a Carathéodory function. $f(x,u)/u$, extended to $u = 0$ by the value 0, is Hölder continuous on subsets where u is bounded, jointly in x and u.

(F2′) $f(x,u) = o(|u|)$ as $u \to 0$, uniformly in x.

(F3′) $|\partial_u f(x,u)| \le a(1 + |u|^{p-2})$ for $u \in \mathbb{R}$ and $x \in \mathbb{R}^N$, with some $p \in (2, 2^*)$.

(F4′) $\partial_u f(x,u)u^2 \ge (\theta - 1)f(x,u)u > 0$ for $u \ne 0$ and $x \in \mathbb{R}^N$, with some $\theta > 2$.

(F5′) $\inf_{x\in\mathbb{R}^N} F(x,1) > 0$.

We define

$$m := \min\Big\{\inf V(\mathbb{R}^N),\ 1\Big\}, \tag{3.2}$$

$$M := \max\Big\{\sup V(\mathbb{R}^N),\ 1\Big\}, \tag{3.3}$$

$$b_1 := \sup\{\, u_0 \ge 0 \mid \forall x \in \mathbb{R}^N\ \forall u \in (0,u_0]\colon\ f(x,u) \le mu/2 \,\}, \tag{3.4}$$

$$b_2 := \inf\{\, u \ge 0 \mid \exists x \in \mathbb{R}^N\colon\ f(x,u) \ge mu \,\}, \tag{3.5}$$

and

$$b_3 = \inf_{x\in\mathbb{R}^N} F(x,1) > 0. \tag{3.6}$$

By the properties of V and f all of these constants are finite and positive. Note that f satisfies the global Ambrosetti-Rabinowitz condition

$$f(x,u)u \geq \theta F(x,u) > 0 \qquad \text{for } u \neq 0 \text{ and } x \in \mathbb{R}^N. \tag{3.7}$$

Integrating this inequality with respect to u over $[1,t]$ yields

$$F(x,t) \geq b_3|t|^\theta \qquad \text{for } |t| \geq 1 \text{ and } x \in \mathbb{R}^N. \tag{3.8}$$

In this setting the energy functional is defined on E as

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x.$$

Again, critical points of J correspond to classical solutions of (3.1). We reuse the notation from Section 1 with respect to sets of critical points of J, but we define c_0 by

$$c_0 := \inf_{\substack{u\in E\setminus\{0\}\\ u\geq 0}} \max_{t>0} J(tu). \tag{3.9}$$

Note that $0 < c_0 < \infty$, that $c_0 \leq \inf J(K_+)$ by (F4′), but that here c_0 is not necessarily a critical level, while in the x-periodic case with f odd in u this definition coincides with that given in (1.2). It is not known under our present conditions whether (3.1) has a nontrivial solution at all.

We adopt the following convention:

(∗) All constants denoted by C_k and D_k, where $k \in \mathbb{N}$, are positive and depend only on m, M, an upper bound for the Hölder norm of V, the data of f, and the extra dependencies given. Moreover, they can be estimated explicitly.

The constants C_k retain their meaning in the whole paper, while the constants D_k retain their meaning only within proofs.

The main purpose of this section is to prove the following more general version of Theorem 1.2:

3.1 Theorem. *Fix $\varepsilon > 0$. Then there are positive constants C_1, C_2, C_3 and C_4 that depend on ε and conform to (∗), with the following property: Given any $u \in K_+^{2c_0-\varepsilon}$ denote by $\mathcal{M}$ the set of local maximum points of u, and denote by x_0 the center of mass of* conv$(\mathcal{M})$. *Then*

$$C_3\mathrm{e}^{-C_1|x-x_0|} \leq u(x)^2 \leq C_4\mathrm{e}^{-C_2|x-x_0|} \tag{3.10}$$

for all $x \in \mathbb{R}^N$.

We introduce the notation $|\cdot|_q$ for the $L^q(\mathbb{R}^N)$-norm if $q \in [1,\infty]$ and prepare the proof of Theorem 3.1 with two technical lemmata:

3.2 Lemma. *There are positive constants C_5, C_6, C_7, C_8, C_9, and C_{10}, $C_9 \le 1$, that conform to $(*)$ and satisfy*

$$C_5 \le c_0 \le C_6, \qquad \|u\| \le C_7 \qquad \text{and} \qquad |u|_\infty \le C_8$$

if $u \in K_+^{2c_0}$. Moreover,

$$\frac{u(x)}{u(y)} \ge C_9 \mathrm{e}^{-C_{10}|x-y|} \tag{3.11}$$

for $u \in K_+^{2c_0}$ and $x, y \in \mathbb{R}^N$.

Proof. We start with exhibiting a lower bound for c_0. Suppose that a nonnegative $u \in E\backslash\{0\}$ satisfies $J(u) = \max_{t>0} J(tu)$. Then $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=1} J(tu) = 0$ implies that

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + V u^2)\,\mathrm{d}x = \int_{\mathbb{R}^N} f(x,u)u\,\mathrm{d}x. \tag{3.12}$$

By (F2′) and (F3′) there is D_1 such that

$$|f(x,u)| \le \frac{m}{2}|u| + D_1|u|^{p-1} \qquad x \in \mathbb{R}^N,\ u \in \mathbb{R}.$$

Therefore

$$\|u\|^2 \le \frac{2}{m}\int_{\mathbb{R}} \Big(|\nabla u|^2 + (V - m/2)u^2\Big)\,\mathrm{d}x \le \frac{2D_1}{m}|u|_p^p.$$

Using the Sobolev embedding $H^1(\mathbb{R}^N) \subseteq L^p(\mathbb{R}^N)$ we obtain D_2 with

$$\|u\| \ge D_2 m^{1/(p-2)}. \tag{3.13}$$

On the other hand it follows from (3.7) and (3.12) that

$$J(u) \ge \left(\frac{1}{2} - \frac{1}{\theta}\right) \int_{\mathbb{R}} \Big(|\nabla u|^2 + V u^2\Big)\,\mathrm{d}x \ge m\left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2 \qquad \text{if } J(u) = \max_{t>0} J(tu). \tag{3.14}$$

Setting

$$C_5 := m^{p/(p-2)} D_2^2 \left(\frac{1}{2} - \frac{1}{\theta}\right),$$

from (3.13) and (3.14) we obtain $J(u) \ge C_5$. The definition of c_0 therefore yields $c_0 \ge C_5$.

To find an upper bound for c_0 define $\varphi\colon \mathbb{R}^N \to \mathbb{R}$ by

$$\varphi(x) := \begin{cases} 1 - |x| & |x| \le 1 \\ 0 & \text{otherwise,} \end{cases}$$

and denote

$$D_3 := \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla\varphi|^2 + M\varphi^2)\,\mathrm{d}x \qquad \text{and} \qquad D_4 := b_3\int_{B_{1/2}(0)}\varphi^\theta\,\mathrm{d}x.$$

Then $F \geq 0$ implies that $J(t\varphi) \leq 4D_3$ for $t \in [0,2]$. For $t \geq 2$ it follows from the definition of b_3 in (3.6), from $\varphi \geq 1/2$ in $B_{1/2}(0)$, and from (3.8) that $J(t\varphi) \leq D_3t^2 - D_4t^\theta$. Taking the definition of c_0 in (3.9) into account we therefore set

$$C_6 := \max\left\{ 4D_3,\ \max_{t\geq 2}(D_3t^2 - D_4t^\theta) \right\}$$

where C_6 depends only on N, M, b_3 and θ. Here we have also used that $\theta > 2$.

In view of (3.14) the definition

$$C_7 := \sqrt{\frac{4\theta C_6}{m(\theta-2)}} \tag{3.15}$$

gives an upper bound for $\|u\|$ if $u \in K_+^{2c_0}$. Standard regularity estimates yield C_8. To compute C_8 from C_7 one could for example use the bootstrapping method outlined in [2, Appendix B], applied to the stationary orbit $u(t) \equiv u$ for the associated parabolic equation. It is easy to see that C_8 can be so chosen that it only depends on N, m, M, a, b_1, b_2, b_3, p and θ.

Finally, the existence of C_9 and C_{10} such that (3.11) holds follows from the upper bound C_8 for $|u|_\infty$, Harnack's inequality as stated in [13, Theorem 8.20], and from the remark immediately following that theorem. □

As is easy to see, (F4′) implies for $u \in E\backslash\{0\}$ that the map $t \mapsto J(tu)$ has a unique positive critical point $\xi(u)$, its maximum point on $[0,\infty)$.

3.3 Lemma. *If $u \in E\backslash\{0\}$ satisfies*

$$|J'(u)u| \leq \frac{\theta-2}{\theta-1}\cdot\frac{m}{2}\|u\|^2 \tag{3.16}$$

then

$$J(u) \geq J(\xi(u)u) - \frac{2|J'(u)u|^2}{m(\theta-2)\|u\|^2}. \tag{3.17}$$

Proof. Define $g(t) := J(tu)$ for $t \geq 0$. Then

$$g'(t) = t\int_{\mathbb{R}^N}(|\nabla u|^2 + Vu^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} f(x,tu)u\,\mathrm{d}x$$

and

$$g''(t) = \int_{\mathbb{R}^N} (|\nabla u|^2 + Vu^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} \partial_u f(x,tu)u^2\,\mathrm{d}x$$
$$\le \frac{\theta-1}{t} g'(t) - (\theta-2)\int_{\mathbb{R}^N} (|\nabla u|^2 + Vu^2)\,\mathrm{d}x$$
$$\le \frac{\theta-1}{t} g'(t) - m(\theta-2)\|u\|^2$$
$$=: h(t), \tag{3.18}$$

where we have used (F4′) and $\theta > 2$.

Note that $g'(t) \le 0$ for $t \ge \xi(u)$. Hence (3.18) implies that $g''(t) \le -m(\theta-2)\|u\|^2$ for all $t \ge \xi(u)$. This yields

$$J'(u)u = g'(1) = \int_{\xi(u)}^{1} g''(s)\,\mathrm{d}s \le -m(\theta-2)\|u\|^2(1-\xi(u)) \le 0 \quad \text{if } \xi(u) \le 1. \tag{3.19}$$

On the other hand, (3.18) and (3.16) imply

$$h(1) = (\theta-1)J'(u)u - m(\theta-2)\|u\|^2 \le \frac{-m(\theta-2)}{2}\|u\|^2 < 0.$$

Moreover, we have

$$h'(t) = -\frac{\theta-1}{t^2}g'(t) + \frac{\theta-1}{t}g''(t) \le -\frac{m(\theta-1)(\theta-2)}{t}\|u\|^2 < 0.$$

Hence $g''(t) \le -m(\theta-2)\|u\|^2/2$ for all $t \ge 1$, and by (3.19)

$$J'(u)u \ge \int_1^{\xi(u)} \frac{m(\theta-2)}{2}\|u\|^2\,\mathrm{d}s = \frac{m(\theta-2)}{2}\|u\|^2(\xi(u)-1) \ge 0 \quad \text{if } \xi(u) \ge 1. \tag{3.20}$$

Combining (3.19) and (3.20) yields

$$|\xi(u) - 1| \le \frac{2|J'(u)u|}{m(\theta-2)\|u\|^2}. \tag{3.21}$$

Observe that by what we have shown above $g''(t) < 0$ for t between 1 and $\xi(u)$. Since $g'(\xi(u)) = 0$ it follows that $|g'(t)| \le |g'(1)| = |J'(u)u|$ for t between 1 and $\xi(u)$. Observing that $J(u) - J(\xi(u)u) = \int_1^{\xi(u)} g'(t)\,\mathrm{d}t$, in conjunction with (3.21) we obtain (3.17). □

Proof of Theorem 3.1. Fix some $u \in K_+^{2c_0-\varepsilon}$ and denote by $\mathcal{M}$ the set of local maximum points of u (recall that $u \in C^2$ by our assumptions on regularity). Then $\mathcal{M} \neq \varnothing$ since $\lim_{|x|\to\infty} u(x) = 0$. Set

$$D_1 := \min\left\{ b_1,\ \frac{b_2 C_9}{2} \right\}.$$

Equation (3.1) and the definition of b_2 in (3.5) imply that

$$u(x) \geq b_2 \qquad \text{if } x \in \mathcal{M}, \tag{3.22}$$

while the definitions of b_1 and D_1 yield

$$f(x,u) \leq \frac{m}{2} u(x) \qquad \text{if } u(x) \leq D_1.$$

Denote

$$A := \{\, x \in \mathbb{R}^N \mid u(x) \leq D_1 \,\} \quad \text{and} \quad \Omega := \mathbb{R}^N \backslash A.$$

Clearly, $A \neq \varnothing$. Moreover, $\Omega \supseteq \mathcal{M} \neq \varnothing$ by (3.22) and since $b_2 > D_1$ (recall that $C_9 \leq 1$). Denote by $\mathcal{U}$ the collection of connected components of Ω. Since Ω is open and bounded, every $U \in \mathcal{U}$ is open, bounded and path connected. Our goal is to estimate the diameter of Ω from above. This easily implies the growth bounds for u, as we will see at the end of the proof.

First we estimate the number of connected components of Ω from above. Fix some $U \in \mathcal{U}$. Then u achieves its maximum on U in some $x_0 \in U$, and by (3.22) and Lemma 3.2 U includes an open ball of radius

$$R := \frac{\log 2}{C_{10}}$$

with center x_0. This follows from

$$u(x) \geq u(x_0) C_9 \mathrm{e}^{-C_{10}|x-x_0|} > \frac{1}{2} b_2 C_9 \geq D_1$$

for $|x - x_0| < R$. Since U was chosen arbitrarily from $\mathcal{U}$, $\|u\| \leq C_7$ implies for $\#\mathcal{U}$, the number of connected components of Ω, that $\#\mathcal{U} |B_R| D_1^2 \leq |u|_2^2 \leq C_7^2$. Here $|B_R|$ denotes the volume of the ball of radius R in $\mathbb{R}^N$. Hence

$$\#\mathcal{U} \leq D_2 \tag{3.23}$$

with

$$D_2 := \left\lfloor \frac{C_7^2}{|B_R| D_1^2} \right\rfloor.$$

Second, we give an upper bound for the diameter of a connected component of Ω. Fix some $U \in \mathcal{U}$ again. For every $x \in U$ it holds by Lemma 3.2 that $u \geq D_1 C_9/2$ on $B_R(x)$. Suppose there exist $x_0, x_1 \in U$ with $|x_0 - x_1| \geq 3R$. Set

$$k := \left\lfloor \frac{|x_0 - x_1|}{3R} \right\rfloor.$$

Then there exist $x_2, x_3, \dots, x_k \in U$ such that

$$B_R(x_i) \cap B_R(x_j) = \varnothing \qquad \text{if } i \neq j, \text{ for } i,j = 0,1,2,\dots,k. \tag{3.24}$$

To see this, assume for simplicity that $x_0 = 0$ and $x_1 = (x_1^1, 0, 0, \dots, 0)$. If $k \geq 2$, choose x_i from the intersection of the hyperplane $\{ x \in \mathbb{R}^N \mid x^1 = 3R(i-1) \}$ with U, for $i = 2, 3, \dots, k$. This intersection is not empty because U is (path-)connected.

It now follows from (3.24) that

$$(k+1)|B_R| \left(\frac{D_1 C_9}{2} \right)^2 \leq |u|_2^2 \leq C_7^2$$

and hence

$$|x_0 - x_1| \leq (k+1) 3R \leq \frac{4C_7^2}{|B_R| D_1^2 C_9^2} 3R.$$

With

$$D_3 := 3R \max \left\{ 1, \frac{4C_7^2}{|B_R| D_1^2 C_9^2} \right\}$$

we obtain

$$\operatorname{diam} U \leq D_3 \qquad \text{for all } U \in \mathcal{U}. \tag{3.25}$$

In the next step we give an upper bound for the distance of connected components of Ω. We fix $U \in \mathcal{U}$ and some $x_0 \in U$, so $U \subseteq B_{D_3}(x_0)$. We want to estimate the maximum distance of $B_{D_3}(x_0)$ from $\Omega \backslash U$. Suppose therefore that

$$\Omega \backslash U \subseteq \mathbb{R}^N \backslash B_{D_3+2r+4}(x_0) \tag{3.26}$$

for some $r \geq 0$. We first prove a decay estimate for u in the annular domain $\Omega' := U_{D_3+2r+4}(x_0) \backslash B_{D_3}(x_0) \subseteq A$. By the definitions of A, D_1, and b_1 we have $u \leq D_1$ and $f(x,u)/u \leq m/2$ in Ω'.

Define μ_1 to be the positive root of the equation

$$\mu^2 + \frac{N-1}{D_3} \mu = \frac{m}{2}$$

and set

$$v(x) := 2D_1 \mathrm{e}^{-\mu_1 (r+2)} \cosh\Big(-\mu_1 (|x - x_0| - D_3 - r - 2) \Big).$$

A straightforward calculation yields $\Delta v = c(x) v$ for $x \neq x_0$ with

$$c(x) := \mu_1^2 - \frac{\mu_1 (N-1)}{|x - x_0|} \tanh\Big(-\mu_1 (|x - x_0| - D_3 - r - 2) \Big).$$

By the choice of μ_1 we have that $c(x) \leq m/2$ for all $x \in \Omega'$. Hence $u, v \geq 0$ implies

$$-\Delta v + \frac{m}{2} v \geq -\Delta v + cv = 0 = -\Delta u + \left(V - \frac{f(x,u)}{u} \right) u \geq -\Delta u + \frac{m}{2} u$$

in Ω'. Since also $v \geq D_1 \geq u$ on $\partial\Omega'$, as a straightforward calculation shows, the maximum principle implies $v \geq u$ on $\overline{\Omega'}$. We therefore obtain:

$$u \leq 2D_1 \mathrm{e}^{-\mu_1(r+2)} \cosh(2\mu_1) = D_{10}\mathrm{e}^{-\mu_1 r} \qquad \text{on } B_{D_3+r+4}(x_0)\backslash B_{D_3+r}(x_0), \tag{3.27}$$

where we have set

$$D_{10} := 2D_1\mathrm{e}^{-2\mu_1}\cosh(2\mu_1) = D_1(1+\mathrm{e}^{-4\mu_1}).$$

Set $\tilde{B}_r := B_{D_3+r+3}(x_0)\backslash B_{D_3+r+1}(x_0)$. The bound $\|u\| \leq C_7$ and regularity theory imply an *a priori* estimate for a global Hölder norm of u. Therefore (F1′) yields an *a priori* estimate for a global Hölder norm of $f(x,u)/u$. If $x \in \partial B_{D_3+r+2}(x_0)$ then $\operatorname{int} B_2(x) \subseteq B_{D_3+r+4}(x_0)\backslash B_{D_3+r}(x_0)$, and hence $u \leq D_{10}\mathrm{e}^{-\mu_1 r}$ on $\operatorname{int} B_2(x)$ by (3.27). Applying [13, Corollary 6.3] with $d = 1$ and the base domain $\operatorname{int} B_2(x)$ to the equation $(-\Delta + V - f(x,u)/u)u = 0$ yields D_9 such that $|\nabla u| \leq D_9\mathrm{e}^{-\mu_1 r}$ on $B_1(x)$. Since the collection of balls $B_1(x)$ with $x \in \partial B_{D_3+r+2}(x_0)$ covers $\tilde{B}_r$ we obtain

$$|\nabla u| \leq D_9\mathrm{e}^{-\mu_1 r} \qquad \text{on } \tilde{B}_r. \tag{3.28}$$

Define a cutoff function $\zeta\colon \mathbb{R} \to [0,1]$ by

$$\zeta(t) := \begin{cases} 0 & s \leq 0, \\ s & 0 \leq s \leq 1, \\ 1 & 1 \leq s. \end{cases}$$

Set

$$u_1(x) := \zeta(D_3 + r + 2 - |x - x_0|)u(x),$$
$$u_2(x) := \zeta(|x - x_0| - D_3 - r - 2)u(x).$$

Then $u_1, u_2 \in E$ are continuous and

$$|\operatorname{supp} u_1 \cap \operatorname{supp} u_2| = 0. \tag{3.29}$$

Moreover, $u_1 = u$ in $B_{D_3+r+1}(x_0)$ and $u_2 = u$ in $\mathbb{R}^N\backslash B_{D_3+r+3}(x_0)$. As noted before there exist $x_1 \in U \cap \mathcal{M}$ and $x_2 \in (\Omega\backslash U) \cap \mathcal{M}$, so $B_R(x_1) \subseteq U$ and $B_R(x_2) \subseteq \Omega\backslash U$. Setting $\delta := |B_R|^{1/2}D_1$ we thus obtain $\|u_i\| \geq \delta$ for $i = 1, 2$.

Define $\bar{u} := u_1 + u_2$. Then $\bar{u} = u$ in $\mathbb{R}^N\backslash\tilde{B}_r$. It holds that

$$0 \leq \bar{u} \leq u$$
$$|u - \bar{u}|^2,\ |u^2 - \bar{u}^2| \leq u^2$$
$$|\nabla u - \nabla\bar{u}|^2,\ \big|\,|\nabla u|^2 - |\nabla\bar{u}|^2\big| \leq 2(|\nabla u|^2 + u^2).$$

Observe that by (F3′)

$$|J(u) - J(\bar{u})| \le \frac{1}{2}\int_{\widetilde{B}_r} (|\,|\nabla u|^2 - |\nabla \bar{u}|^2| + M|u^2 - \bar{u}^2|)\,\mathrm{d}x + 2a\int_{\widetilde{B}_r}\left(\frac{|u|^2}{2} + \frac{|u|^p}{p(p-1)}\right)\mathrm{d}x.$$

Using (3.27), and (3.28) we may therefore choose a function $g_1\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ that is strictly decreasing, that satisfies $g_1(r) \to 0$ as $r \to \infty$, that depends only on the parameters D_9, D_{10}, μ_1, M, a and p, and that satisfies

$$|J(u) - J(\bar{u})| \le g_1(r). \tag{3.30}$$

We choose a function g_2 with similar properties as g_1 that satisfies

$$\|J'(u) - J'(\bar{u})\| \le g_2(r) \tag{3.31}$$

instead of (3.30). Then $J'(u) = 0$, (3.29) and (3.31) imply

$$|J'(u_i)u_i| = |J'(\bar{u})u_i| \le g_2(r)\|u_i\|. \tag{3.32}$$

If for $i = 1$ or $i = 2$

$$|J'(u_i)u_i| > \frac{\theta - 2}{\theta - 1}\cdot\frac{m\|u_i\|^2}{2}$$

holds then by (3.32)

$$g_2(r) \ge \frac{\theta - 2}{\theta - 1}\cdot\frac{m\delta}{2},$$

respectively

$$r \le g_2^{-1}\left(\frac{\theta - 2}{\theta - 1}\cdot\frac{m\delta}{2}\right) \tag{3.33}$$

since g_2 is strictly decreasing in r.

Recall that by the definitions of c_0 in (3.9) and ξ just before Lemma 3.3 any nonnegative $u \in E\backslash\{0\}$ satisfies $J(\xi(u)u) \ge c_0$. If for $i = 1, 2$

$$|J'(u_i)u_i| \le \frac{\theta - 2}{\theta - 1}\cdot\frac{m\|u_i\|^2}{2}$$

holds, this fact, Lemma 3.3, (3.30) and (3.32) imply

$$\begin{aligned} 2c_0 - \varepsilon \ge J(u) &\ge J(\bar{u}) - g_1(r) \\ &= J(u_1) + J(u_2) - g_1(r) \\ &\ge J(\xi(u_1)u_1) + J(\xi(u_2)u_2) - g_1(r) - \frac{4}{m(\theta - 2)}g_2(r)^2 \\ &\ge 2c_0 - g_3(r), \end{aligned}$$

where we have set

$$g_3(r) := g_1(r) + \frac{4}{m(\theta-2)} g_2(r)^2.$$

Hence

$$r \le g_3^{-1}(\varepsilon). \tag{3.34}$$

Since $U \in \mathcal{U}$ and $x_0 \in U$ were chosen arbitrarily, setting

$$D_6 := 4 + 2\max\left(\{0\} \cup g_2^{-1}\left(\frac{\theta-2}{\theta-1}\cdot\frac{m\delta}{2}\right) \cup g_3^{-1}(\varepsilon)\right), \tag{3.35}$$

and taking (3.26), (3.33), and (3.34) into account we obtain

$$\operatorname{dist}(B_{D_3}(x), \Omega\backslash U) \le D_6 \qquad \text{for every } U \in \mathcal{U} \text{ and every } x \in U. \tag{3.36}$$

We can now conclude easily. Recall that by (3.25) every $U \in \mathcal{U}$ is contained in a ball of diameter $2D_3$. Combining this fact with (3.23) and (3.36) we see that $\operatorname{diam}(\Omega) \le D_7$, with $D_7 := 2D_2D_3 + (D_2 - 1)D_6$. Hence we obtain $\Omega \subseteq B_{D_7}(x)$ for all $x \in \Omega$. Moreover, if x_0 is the center of mass of $\operatorname{conv}(\mathcal{M})$ then $x_0 \in B_{D_7}(x)$ for all $x \in \Omega$. Therefore

$$\Omega \subseteq B_{D_7}(x_0). \tag{3.37}$$

Pick any $x_1 \in \mathcal{M} \subseteq \Omega$. By Lemma 3.2 and (3.37) every $x \in \mathbb{R}^N$ satisfies

$$u(x)^2 \ge (b_2C_9)^2 \mathrm{e}^{-2C_{10}|x-x_1|} \ge (b_2C_9)^2\mathrm{e}^{-2C_{10}|x_0-x_1|}\mathrm{e}^{-2C_{10}|x-x_0|} \ge C_3\mathrm{e}^{-C_1|x-x_0|}$$

with $C_1 := 2C_{10}$ and $C_3 := (b_2C_9)^2\mathrm{e}^{-C_1D_7}$.

On the other hand, by (3.37) and the maximum principle it follows as in the proof of (3.27) that $u(x) \le D_1 \exp(-\mu_2(|x-x_0| - D_7))$ for $x \in \mathbb{R}^N\backslash B_{D_7}(x_0)$, with $\mu_2 := \sqrt{m/2}$. Recall that $C_8 \ge b_3 \ge D_1$. Setting $C_4 := C_8^2 \exp(2\mu_2 D_7)$ and $C_2 := 2\mu_2$ it follows that $u(x)^2 \le C_4 \exp(-C_2|x-x_0|)$ for all $x \in \mathbb{R}^N$. □

3.4 Remark. A similar estimate can be proved for $u \in K_-$. Instead of (F5′) one has to assume that $\inf_{x\in\mathbb{R}^N} F(x,-1) > 0$ and adapt the definitions of c_0, b_1, b_2, and b_3 accordingly.

3.5 Remark. Condition (F1′) could be changed by assuming Hölder continuity for f instead of $f(x,u)/u$ on sets where u is bounded, at the cost of more involved dependencies in the constants (see the proof of Eq. (3.28)).

3.6 Remark. The mere existence of constants C_1, C_2, C_3 and C_4 such that (3.10) holds for all $u \in K_+^{2c_0-\varepsilon}$ can be proved under weaker assumptions on f if f and V are periodic in x. Namely, instead of (F1′) it suffices to assume that f is Hölder continuous on subsets where u is bounded, assumption (F3′) can be replaced by

$$|f(x,u)| \le a(1+|u|^{p-1}) \qquad \text{for } u \in \mathbb{R} \text{ and } x \in \mathbb{R}^N,$$

and (F4′) can be replaced by the global Ambrosetti-Rabinowitz condition

$$f(x,u)u \geq \theta F(x,u) > 0 \qquad \text{for } u \neq 0 \text{ and } x \in \mathbb{R}^N.$$

Condition (F5′) is now a consequence of the assumptions above.

In this setting one defines c_0 by (2.2), and recycles the definitions of b_1, b_2, and b_3 from (3.4), (3.5), and (3.6). Suppose that (u_n) is a sequence in $K_+^{2c_0-\varepsilon}$, and suppose that (x_n) is a sequence in $\mathbb{R}^N$ such that each x_n is a local maximum point for u_n. Assume that there is a sequence (y_n) in $\mathbb{R}^N$ such that $u(y_n) > b_1$ for each n and $|x_n - y_n| \to \infty$ as $n \to \infty$. We have $u_n(x_n) \geq b_2$ for all n. Note that Lemma 3.2 holds under the present weaker assumptions. Using concentration compactness arguments (see [3, Proposition 2.5]) and (3.11) we reach a contradiction, since the energy $J(u_n)$ remains bounded by $2c_0 - \varepsilon$. Therefore there exists $R > 0$ such that $u_n(x) \leq b_1$ and hence $f(x, u_n(x))/u_n(x) \leq m/2$ if $|x - x_n| \geq R$. It is easy to conclude from here. But note that this proof, being nonconstructive in nature, does not yield explicit estimates of the constants.

Proof of Theorem 1.3. We fix the constants $m := \min W$, $M := \max W$, and an upper bound for the Hölder norms of V_T, which applies to V_T as long as $T \geq I$. Also fixing $\varepsilon := 2c_0 - c$ Theorem 3.1 yields constants C_1, C_2, C_3, and C_4 with the following property: If $T \geq I$, and if $u \in K_+^c$ (with V_T in place of V), then, denoting by $\mathcal{M}$ the set of local maximum points of u and by x_0 the center of mass of $\operatorname{conv}(\mathcal{M})$, Eq. (3.10) holds. If in addition u is even in x^i for some $i \in \{1, 2, \ldots, N\}$ then $x_0^i = 0$. It follows that $(\mathrm{I})_{c_0}$ is satisfied for T large enough, since as $T \to \infty$ all $u \in K_+^c$ (with V_T in place of V) that are even in x^i remain concentrated near $\{x^i = 0\}$, where $\partial_i^2 V_T$ is negative. □

4. An Example in Dimension One

In this section we explain how to prove numerically the validity of $(\mathrm{S})_{c_0}$ for V as given in Example 1.5 and for $p = 20$. More generally, we will consider p as a parameter. Recall the 1-dimensional problem

$$-u'' + Vu = |u|^{p-2}u, \qquad u \in H^1(\mathbb{R}^N). \tag{4.1}$$

To facilitate the presentation we say that V satisfying (V1)–(V4) is *p-admissible* if $(\mathrm{S})_{c_0}$ holds for (4.1) with some $p > 2$.

Given $M > 1$ we specialize Theorem 1.3 to dimension one and to the specific function W with period 1, defined by

$$\min W = 1, \quad W'(0) = 0,$$

and

$$W''(x) = \begin{cases} -d, & \text{if } x \in k + [-1/4, 1/4] \text{ for some } k \in \mathbb{Z}, \\ d, & \text{otherwise,} \end{cases}$$

where $d := 16(M-1)$. Then $W \in C^1(\mathbb{R},\mathbb{R})$, W' is Lipschitz continuous, W'' exists classically and is negative in $(-1/4, 1/4)$, $\max W = M$, and W is even.

In this setting the constant $\tau_0 = \tau_0(M,p)$ is, by definition, the only element of T_0 from Theorem 1.3. Hence, writing $V_\tau(x) := W(x/\tau)$ for $\tau > 0$ and $x \in \mathbb{R}$, τ_0 is such that

$$-u'' + V_\tau(x)u = |u|^{p-2}u, \qquad u \in H^1(\mathbb{R}^N), \tag{4.2}$$

satisfies $(\mathrm{I})_{c_0}$, and therefore is p-admissible by Theorem 1.1, for $\tau \geq \tau_0$. As we will show below, this minimal period τ_0 can be optimized and estimated in an elementary manner in terms of the parameters p and M, taking advantage of the simpler geometry in $\mathbb{R}$ (as opposed to $\mathbb{R}^N$ with $N \geq 2$).

At the end of this section we can construct the potential V of Example 1.5 by rescaling a given $V_{\tau_0(M,20)}$, where M is appropriately chosen.

This section should be read in conjunction with Section 3 since we just mention the differences, and we also rely on notation introduced there. We define c_0 by (3.9) and note that it coincides with the definitions in (1.2) and (2.2).

4.1. Preliminary Estimates

Here we establish various bounds that were not calculated explicitly in Section 3 for the general case. Note that in the present situation $b_1 = 2^{-1/(p-2)}$, $b_2 = 1$, and $b_3 = 1/p$.

4.1.1. Sobolev Constants and Gradient Estimates

For an open interval $\Omega := (-l, l)$, $0 < l \leq \infty$, we have an embedding of $H^1(\Omega)$ into the space of bounded Lipschitz continuous functions on Ω. In a simple way we derive upper bounds for the norms of the embeddings of $H^1(\Omega)$ into $L^q(\Omega)$, $q \in [2,\infty]$. These techniques are of course well known. We only provide the proofs here for the convenience of the reader, and since we are interested in explicit estimates.

Consider $u \in H^1(\Omega)$ and choose some $x \in [0, l)$. Then for all $y \in (-l, x]$ we have

$$u(x) = \int_y^x u'(s)\,\mathrm{d}s + u(y)$$

and hence

$$|u(x)| \leq \sqrt{x-y}\,|u'|_2 + |u(y)| \tag{4.3}$$

by Hölder's inequality. For $z \in [-l, x]$ integrate (4.3) over (z, x) with respect to y and obtain, after using Hölder's inequality again and dividing by $|x-z|$, that

$$\begin{aligned} |u(x)| &\leq \left(\frac{2}{3}\sqrt{x-z}\,|u'|_2 + \frac{1}{\sqrt{x-z}}|u|_2\right) \\ &\leq \left(\frac{4}{9}|x-z| + \frac{1}{|x-z|}\right)^{1/2} \|u\|. \end{aligned}$$

The last term in the above expression is minimized by choosing z in such a way that $|x-z| = \min\{\,3/2,\ l\,\}$. In the same way we treat the case of $x \in (-l, 0]$. Setting

$$C_{\mathrm{S}}(l,\infty) := \left(\frac{4}{9}\min\{\,3/2,\ l\,\} + \frac{1}{\min\{\,3/2,\ l\,\}}\right)^{1/2}$$

we thus obtain $|u|_\infty \le C_{\mathrm{S}}(l,\infty)\|u\|$ for all $u \in H^1(\Omega)$. Note that $C_{\mathrm{S}}(l,\infty) = C_{\mathrm{S}}(3/2,\infty)$ for $l \ge 3/2$. Therefore $C_{\mathrm{S}}(\infty,\infty) := C_{\mathrm{S}}(3/2,\infty)$ satisfies $|u|_\infty \le C_{\mathrm{S}}(\infty,\infty)\|u\|$ for all $u \in H^1(\mathbb{R})$. We also define

$$C_{\mathrm{S}}(l,p) := C_{\mathrm{S}}(l,\infty)^{\frac{p-2}{p}}$$

for $p \ge 2$. Then $|u|_p \le C_{\mathrm{S}}(l,p)\|u\|$ for all $u \in H^1(\Omega)$ by interpolation.

Now consider a positive solution u of (4.2). We want to give a pointwise estimate of u' in $[-l,l]$ (some $l > 0$) in terms of an upper bound on $u(x)$ for $x \in [-l,l]$, assuming that $u \le 1$ on $[-l,l]$. Therefore fix $x \in [-l,l]$ and choose $y \in [-l,l]$ such that $|x-y| = l$. Recall that $u \in C^2$ since V_τ is differentiable. There is z between x and y such that

$$u'(x) = \frac{u(y)-u(x)}{y-x} - \frac{1}{2}u''(z)(y-x).$$

It follows from (4.2) that

$$|u'(x)| \le \frac{1}{l}|u(x)-u(y)| + \frac{M}{2}|u(z)|l \le \max_{s\in[-l,l]}|u(s)|\left(\frac{1}{l} + \frac{M}{2}l\right)$$

(here we have used that $u \ge 0$). Since x was chosen arbitrarily from $[-l,l]$ we obtain

$$\max_{x\in[-l,l]}|u'(x)| \le \max_{s\in[-l,l]}|u(s)|\left(\frac{1}{l} + \frac{M}{2}l\right) \qquad \text{if } u \text{ solves (4.2), } 0 \le u \le 1 \text{ on } [-l,l]. \tag{4.4}$$

4.1.2. Bounds on c_0 and Their Consequences

To obtain a lower bound for c_0 assume that $u \in K_+^{c_0}$ and consider

$$\|u\|^2 \le \int_{\mathbb{R}}\left(|u'|^2 + V_\tau u^2\right)\mathrm{d}x = \int_{\mathbb{R}} u^p\,\mathrm{d}x \le C_{\mathrm{S}}(\infty,p)^p\|u\|^p.$$

It follows that

$$\|u\| \ge \left(\frac{1}{C_{\mathrm{S}}(\infty,p)}\right)^{\frac{p}{p-2}} = \frac{\sqrt{3}}{2}$$

by the definition of $C_{\mathrm{S}}(\infty,p)$. Therefore we obtain

$$c_0 = J(u) = \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbb{R}}\left(|u'|^2 + V_\tau u^2\right)\mathrm{d}x \ge \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 \ge \frac{3}{8}\cdot\frac{p-2}{p} =: C_5.$$

We estimate c_0 from above similarly as in Lemma 3.2. Being more careful though, we try to get a better estimate by optimizing over a class of functions in $H^1(\mathbb{R})$. Namely, fixing $\varphi(x) = \mathrm{e}^{-x^2}$ we define the class $\{\varphi_\sigma\}_{\sigma>0}$ by setting

$$\varphi_\sigma(x) := \varphi(\sigma x)$$

for $x \in \mathbb{R}$. Set

$$D_1 := \int_{\mathbb{R}} |\varphi'|^2 \,\mathrm{d}x = \sqrt{\frac{\pi}{2}} \qquad \text{and} \qquad D_2(q) := \int_{\mathbb{R}} \varphi^q \,\mathrm{d}x = \sqrt{\frac{\pi}{q}}$$

for $q \geq 2$. A straightforward calculation yields

$$\max_{t>0} J(t\varphi_\sigma) = \left(\frac{a(\sigma)^p}{b(\sigma)^2}\right)^{\frac{1}{p-2}} \left(\frac{2}{p}\right)^{\frac{2}{p-2}} \frac{p-2}{p} \tag{4.5}$$

with

$$\begin{aligned} a(\sigma) &:= \frac{1}{2}\left(D_1\sigma + \frac{MD_2(2)}{\sigma}\right) \\ b(\sigma) &:= \frac{D_2(p)}{p\sigma}. \end{aligned}$$

The expression on the right of (4.5) attains its minimum in

$$\sigma_{\min} := \left(\frac{M(p-2)D_2(2)}{(p+2)D_1}\right)^{1/2} = \sqrt{M\frac{p-2}{p+2}}$$

and we obtain

$$\begin{aligned} a(\sigma_{\min}) &= \frac{1}{2}\sqrt{\frac{\pi M}{2}}\left(\sqrt{\frac{p-2}{p+2}} + \sqrt{\frac{p+2}{p-2}}\right) \\ b(\sigma_{\min}) &= \frac{1}{p}\sqrt{\frac{\pi(p+2)}{pM(p-2)}}. \end{aligned}$$

Therefore we obtain a good upper bound C_6 for c_0 by setting

$$C_6 := \min_{\sigma>0} \max_{t>0} J(t\varphi_\sigma) = \left(\frac{a(\sigma_{\min})^p}{b(\sigma_{\min})^2}\right)^{\frac{1}{p-2}} \left(\frac{2}{p}\right)^{\frac{2}{p-2}} \frac{p-2}{p}. \tag{4.6}$$

As in (3.15) (here with $J(u) \leq c_0$) we have

$$\|u\| \leq \left(\frac{2pC_6}{p-2}\right)^{1/2} =: C_7.$$

Last but not least, using the definition of $C_{\mathrm{S}}(\infty,\infty)$, we set

$$C_8 := C_{\mathrm{S}}(\infty,\infty)C_7 = \frac{2}{3}\sqrt{3}\,C_7.$$

4.1.3. A Harnack Inequality

Our goal here is to provide an inequality as in (3.11). Suppose therefore that $u \in K_+$ and set $v := u'/u$. We claim that

$$|v| \le \sqrt{M} \qquad \text{on } \mathbb{R}. \tag{4.7}$$

Once this claim is proved it is clear that we may set $C_9 = 1$ and $C_{10} = \sqrt{M}$.

For large $|x|$ the function u is the solution of a small perturbation of Hill's Equation

$$-w'' + V_\tau w = 0. \tag{4.8}$$

since $u(x) \to 0$ as $|x| \to \infty$. Suppose that w_1 and w_2 are solutions of (4.8) satisfying $w_1(0) = w_2'(0) = 1$ and $w_1'(0) = w_2(0) = 0$. Since $V_\tau \ge 0$ the functions w_i are convex where $w_i \ge 0$. Hence w_1 and w_2' are increasing in $[0, \infty)$ and consequently

$$w_1(\tau) + w_2'(\tau) \ge w_1(0) + w_2'(0) = 2.$$

Applying standard Floquet theory, cf. [17, Sections 1.2 and 1.3], this inequality implies that there are $\alpha > 0$ and a positive τ-periodic function $r(x)$ such that $\mathrm{e}^{-\alpha x} r(x)$ and $\mathrm{e}^{\alpha x} r(-x)$ form a fundamental system for (4.8). These facts imply that w'/w is bounded for every positive solution w of (4.8). From this one can show that also $v = u'/u$ remains bounded as $|x| \to \infty$ by a perturbation argument (see eg. the discussion in [2, Appendix A.3]).

From Eq. (4.2) we obtain

$$v' = V_\tau - u^{p-2} - v^2.$$

Note that $V_\tau - u^{p-2} \le M$. Together with the boundedness of $v(x)$ as $|x| \to \infty$ these facts imply (4.7).

4.2. Estimating Minimal Periods

In this subsection we present a recipe to numerically calculate $\tau_0 > 0$ such that for every $\tau \ge \tau_0$ and every even $u \in K_+^{c_0}$, a solution of (4.2), it holds that

$$\int_{[-\tau/4, \tau/4]} u^2 \,\mathrm{d}x \ge \int_{\mathbb{R} \setminus [-\tau/4, \tau/4]} u^2 \,\mathrm{d}x. \tag{4.9}$$

By the definition of V_τ this implies that

$$\int_{\mathbb{R}} u^2 V_\tau'' \,\mathrm{d}x \le 0$$

for every such u, that is, $(\mathrm{I})_{c_0}$ holds and V_τ is p-admissible by Theorem 1.1.

Take $D_1 \in (0, 1)$ as a parameter to be optimized at the end. We will find $\tau_1(D_1)$ such that (4.9) holds if $\tau \ge \tau_1(D_1)$, and we set

$$\tau_0 := \inf_{D_1 \in (0,1)} \tau_1(D_1). \tag{4.10}$$

Therefore fix D_1, $\tau > 0$ and an even $u \in K_+^{c_0}$. We will calculate bounds for both sides of the inequality in (4.9) in terms of τ. From these we will derive the minimum period $\tau_1(D_1)$ such that (4.9) holds. Define

$$A := \{\, x \in \mathbb{R}^N \mid u(x) \le D_1 \,\} \quad \text{and} \quad \Omega := \mathbb{R}^N \backslash A$$

as in the proof of Theorem 3.1, denote by $\mathcal{M}$ the set of local maximum points of u, and by $\mathcal{U}$ the set of components of Ω.

As a first step we build $g_4\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that if $U \in \mathcal{U}$ then $\|u\|_U^2 \ge g_4(\operatorname{diam} U)$. To this end fix $U \in \mathcal{U}$. Since $U \cap \mathcal{M} \neq \varnothing$ we pick $x_0 \in U \cap \mathcal{M}$ and note that $u(x_0) \ge 1$. Setting $R := -(\log D_1)/C_{10}$, from (3.11) it follows that $I := (x_0 - R, x_0 + R) \subseteq \Omega$, i.e. $\operatorname{diam} U \ge 2R$. Suppose that $U = (x_1, x_2)$. For a measurable subset A of $\mathbb{R}$ and $q \in [1, \infty]$ we denote by $|\cdot|_{q,A}$ the $L^q(A)$-norm. It follows from (3.11) that

$$|u|_{2,I}^2 \ge \int_{x_0-R}^{x_0+R} \mathrm{e}^{-2C_{10}|x-x_0|}\,\mathrm{d}x = \frac{1-D_1^2}{C_{10}}. \tag{4.11}$$

On the other hand, setting $t = x_2 - x_1$ we obtain

$$|u|_{2,U\backslash I}^2 \ge (t - 2R)D_1^2. \tag{4.12}$$

To estimate $|u'|_{2,U}^2$, note that $u(x_1) = D_1$ because x_1 lies on the boundary of Ω, and consider

$$1 - D_1 \le u(x_0) - u(x_1) = \int_{x_1}^{x_0} u'(x)\,\mathrm{d}x \le \sqrt{x_0 - x_1}\,|u'|_{2,(x_1,x_0)}.$$

Together with a similar inequality on (x_0, x_2) we obtain

$$|u'|_{2,(x_1,x_0)}^2 \ge \frac{(1-D_1)^2}{x_0 - x_1} \qquad \text{and} \qquad |u'|_{2,(x_0,x_2)}^2 \ge \frac{(1-D_1)^2}{x_2 - x_0}$$

and hence

$$|u'|_{2,U}^2 \ge (1-D_1)^2 \left(\frac{1}{x_0 - x_1} + \frac{1}{x_2 - x_0}\right) \ge \frac{4}{t}(1-D_1)^2. \tag{4.13}$$

In view of (4.11), (4.12) and (4.13) we define

$$g_4(t) := \frac{1-D_1^2}{C_{10}} + (t-2R)D_1^2 + \frac{4}{t}(1-D_1)^2 \le |u|_{2,U}^2 + |u'|_{2,U}^2 = \|u\|_U^2. \tag{4.14}$$

Second we estimate the number of connected components of Ω from above. The function g_4 defined above attains its minimum on $[2R, \infty)$ at

$$t_0 := \frac{2(1-D_1)}{D_1}, \tag{4.15}$$

with the value

$$g_4(t_0) = \frac{1 - D_1^2}{C_{10}} + 2D_1^2\left(\frac{1 - D_1}{D_1} - R\right) + 2D_1(1 - D_1)$$

(it is easy to see that $R \le (1 - D_1)/D_1$, so $t_0 \ge 2R$). Since $g_4(t_0)$ is the lowest possible value of $\|u\|_U^2$, we set

$$D_2 := \left\lfloor \frac{C_7^2}{g_4(t_0)} \right\rfloor$$

and obtain

$$\#\mathcal{U} \le D_2$$

as in (3.23).

In the next step we find an upper bound D_6 for the length of an interval separating two adjacent connected components of Ω. To prove exponential decay of u in A in terms of the distance from Ω, note that $u \le D_1$ and $V_\tau - |u|^{p-2} \ge 1 - D_1^{p-2}$ on A. Therefore set

$$\mu := \sqrt{1 - D_1^{p-2}}.$$

Suppose that $[x_1, x_2]$ is a bounded component of A, and that there is $r \ge 0$ such that $t = x_2 - x_1 = 2(r + \beta)$ with $\beta := \sqrt{2/M}$. Setting $x_0 := (x_1 + x_2)/2$, the maximum principle implies as in the proof of (3.27) that

$$u(x) \le 2D_1 \mathrm{e}^{-\mu(r+\beta)} \cosh(\mu(x - x_0))$$

for $x \in [x_1, x_2]$. With

$$\widetilde{B}_r := [x_0 - \beta, x_0 + \beta],$$
$$D_{10} := 2D_1 \mathrm{e}^{-\mu\beta} \cosh(\mu\beta)$$

we obtain

$$u(x) \le D_{10} \mathrm{e}^{-\mu r} \qquad \text{for } x \in \widetilde{B}_r. \tag{4.16}$$

From (4.4) it follows that

$$|u'(x)| \le \sqrt{2M} D_{10} \mathrm{e}^{-\mu r} \qquad \text{for } x \in \widetilde{B}_r. \tag{4.17}$$

Set

$$u_1(x) := \zeta\left(\frac{x_0 - x}{\beta}\right) u(x),$$
$$u_2(x) := \zeta\left(\frac{x - x_0}{\beta}\right) u(x),$$
$$\bar{u} := u_1 + u_2,$$

where ζ is defined as in the proof of Theorem 3.1. Then

$$0 \le \bar{u} \le u$$
$$|u-\bar{u}|^2,\ |u^2-\bar{u}^2| \le u^2$$
$$|u'-\bar{u}'|^2,\ \big|\,|u'|^2-|\bar{u}'|^2\big| \le 2\left(|u'|^2+\frac{u^2}{\beta^2}\right) = (2|u'|^2+Mu^2).$$

Hence (4.16) and (4.17) imply

$$\begin{aligned}|J(u)-J(\bar{u})| &\le \frac{1}{2}\int_{\widetilde{B}_r}\Big(\big|\,|u'|^2-|\bar{u}'|^2\big|+V|u^2-\bar{u}^2|\Big)\,\mathrm{d}x+\frac{2}{p}\int_{\widetilde{B}_r}|u|^p\,\mathrm{d}x\\ &\le 6\sqrt{2M}D_{10}^2\mathrm{e}^{-2\mu r}+\frac{4}{p}\sqrt{\frac{2}{M}}D_{10}^p\mathrm{e}^{-p\mu r}\\ &=: g_1(r).\end{aligned}$$

Similarly, a straightforward calculation yields

$$\begin{aligned}\|J'(u)-J'(\bar{u})\| \le{}& 2^{5/4}\sqrt{3}M^{3/4}D_{10}\mathrm{e}^{-\mu r}\\ &+2^{1+\frac{3(p-1)}{2p}}M^{-\frac{p-1}{2p}}C_{\mathrm{S}}(\beta,p)D_{10}^{p-1}\mathrm{e}^{-(p-1)\mu r} =: g_2(r).\end{aligned}$$

As before we set

$$g_3(r) := g_1(r)+\frac{4}{p-2}g_2(r)^2.$$

Recall the definition of g_4 in (4.14). We use $\delta := \sqrt{g_4(t_0)}$ as a lower bound for $\|u_i\|$ $(i=1,2)$ and follow the argument leading up to the definition of D_6 in (3.35). Here we replace ε by $c_0 = 2c_0 - c_0$ (since we are proving $(\mathrm{I})_{c_0}$), and in turn we replace c_0 by the known *a priori* lower bound $C_5 = 3(p-2)/(8p)$ for c_0, using that g_3^{-1} is monotone decreasing. We therefore arrive at

$$x_2 - x_1 \le D_6 \qquad \text{if } [x_1,x_2] \text{ is a bounded connected component of } A,$$

where

$$D_6 := 2\beta + 2\max\left(\{0\}\cup g_2^{-1}\left(\frac{\delta(p-2)}{2(p-1)}\right)\cup g_3^{-1}\left(\frac{3(p-2)}{8p}\right)\right).$$

Instead of globally estimating $\operatorname{diam}\mathcal{U}$ as in the proof of Theorem 3.1 we take a different approach here, utilizing the simpler geometry in one dimension, and carefully retaining accurate estimates. We have the upper bound D_2 for $\#\mathcal{U}$. For our specific class of potentials V_τ and for $Z \in \{1,2,\ldots,D_2\}$ we calculate $\tau_2(D_1,Z)$ such that (4.9) holds if $\tau \ge \tau_2(D_1,Z)$ and $\#\mathcal{U} = Z$. Note that all estimates up to now were independent of τ, even though we employed the periodicity of the potential V_τ in Section 4.1.3. We then take

$$\tau_1(D_1) := \max_{Z\in\{1,2,\ldots,D_2\}} \tau_2(D_1,Z), \tag{4.18}$$

so (4.9) is satisfied if $\tau \geq \tau_1(D_1)$, independently of $\#\mathcal{U}$.

Therefore, fix $Z \in \{1, 2, \ldots, D_2\}$ for now, suppose that $\mathcal{U} = \{ U_1, U_2, \ldots, U_Z \}$, and set $t_i := |U_i| = \operatorname{diam} U_i$. Then $|\Omega| = \sum_{i=1}^{Z} t_i$. To obtain an upper estimate for $|U|$ note that

$$C_7^2 \geq \sum_{i=1}^{Z} \|u\|_{U_i}^2 \geq \sum_{i=1}^{Z} g_4(t_i).$$

Using the properties of g_4 it is elementary to show that the function $(t_1, t_2, \ldots, t_Z) \mapsto \sum_{i=1}^{Z} t_i$ attains its maximum under the side conditions

$$\sum_{i=1}^{Z} g_4(t_i) \leq C_7^2 \qquad \text{and} \qquad \forall i\colon t_i \geq 2R$$

in a point $(t_1, t_2, \ldots, t_Z)$ with $t_0 \leq t_1 = t_2 = \cdots = t_Z =: t_{\max}$ and

$$\sum_{i=1}^{Z} g_4(t_i) = Z g_4(t_{\max}) = C_7^2$$

(recall that t_0 is defined in (4.15) and that $Z g_4(t_0) \leq C_7^2$). Hence

$$t_{\max} = (g_4|_{[t_0,\infty)})^{-1} \left(\frac{C_7^2}{Z} \right) \qquad \text{and} \qquad |\Omega| \leq Z t_{\max} =: D_{11}.$$

We therefore set $D_7 := D_{11} + (Z-1) D_6$. Then $\Omega \subseteq [-D_7/2, D_7/2]$ because u is even.

Recall that $|u|_{2,U}^2 \geq (1 - D_1^2)/C_{10}$ for $U \in \mathcal{U}$ by (4.11). Suppose that $\tau \geq 2D_7$. Then

$$\int_{-\tau/4}^{\tau/4} u^2 \, \mathrm{d}x \geq \frac{Z(1 - D_1^2)}{C_{10}}.$$

On the other hand,

$$u(x) \leq D_1 \mathrm{e}^{-\mu(x - D_7/2)} \qquad \text{for } x \geq \frac{D_7}{2},$$

so

$$\int_{\mathbb{R} \setminus [-\tau/4, \tau/4]} u^2 \, \mathrm{d}x \leq 2 \int_{\tau/4}^{\infty} D_1^2 \mathrm{e}^{-2\mu(x - D_7/2)} \, \mathrm{d}x = \frac{D_1^2}{\mu} \mathrm{e}^{-\mu(\tau/2 - D_7)}.$$

To achieve (4.9) we therefore require that

$$\frac{Z(1 - D_1^2)}{C_{10}} \geq \frac{D_1^2}{\mu} \mathrm{e}^{-\mu(\tau/2 - D_7)}$$

respectively that

$$\tau \geq 2 \left(D_7 - \frac{1}{\mu} \log \left(\frac{\mu Z (1 - D_1^2)}{C_{10} D_1^2} \right) \right).$$

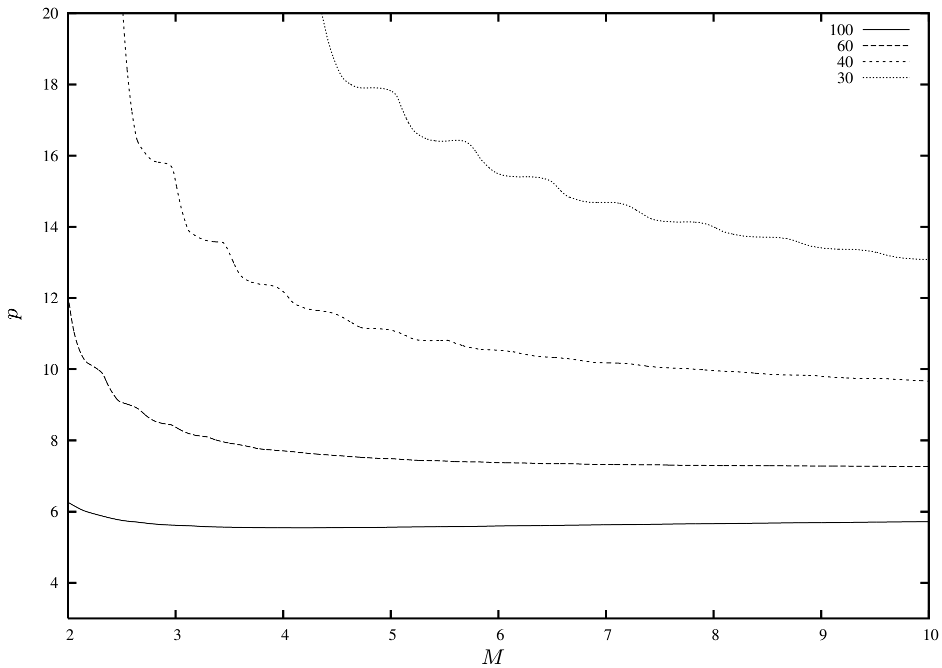

Figure 2: Isolines of σ_0 for small values of p and M

We therefore set

$$\tau_2(D_1, Z) := 2D_7 + \max\left\{0,\ \frac{2}{\mu}\log\left(\frac{C_{10}D_1^2}{\mu Z(1-D_1^2)}\right)\right\}.$$

Taking (4.18) and (4.10) into account the recipe for numerically calculating $\tau_0 = \tau_0(M,p)$ is complete.

4.1 Remark. The definition of C_6 in (4.6) yields that $C_6 = C(p) \cdot M^{\frac{p+2}{2(p-2)}}$ with some positive constant $C(p)$. Following the dependencies on large M throughout the estimates above, for τ_0 as defined in (4.10) we obtain that

$$\lim_{M\to\infty} \frac{\tau_0}{M^{\frac{p+2}{2(p-2)}}\log M} \qquad \text{exists for fixed } p \text{ and is positive.} \tag{4.19}$$

4.3. Numerical Justification of Example 1.5

To measure the "reasonability" of V_{τ_0} we introduce the ratio $\sigma_0(M,p) := \tau_0(M,p)/(M-1)$. Since sufficiently large periods τ always make V_τ p-admissible, we strive to find not too large M and p such that the corresponding $\sigma_0(M,p)$ is reasonably small. Evaluating the recipe of the previous section numerically we present plots of isolines of the function σ_0 in Figs. 2 and 3. Note that $\lim_{M\to\infty}\sigma_0(M,p) = \infty$ if $p \le 6$ and $\lim_{M\to\infty}\sigma_0(M,p) = 0$ if $p > 6$. This can be explained by the asymptotic estimate in (4.19).

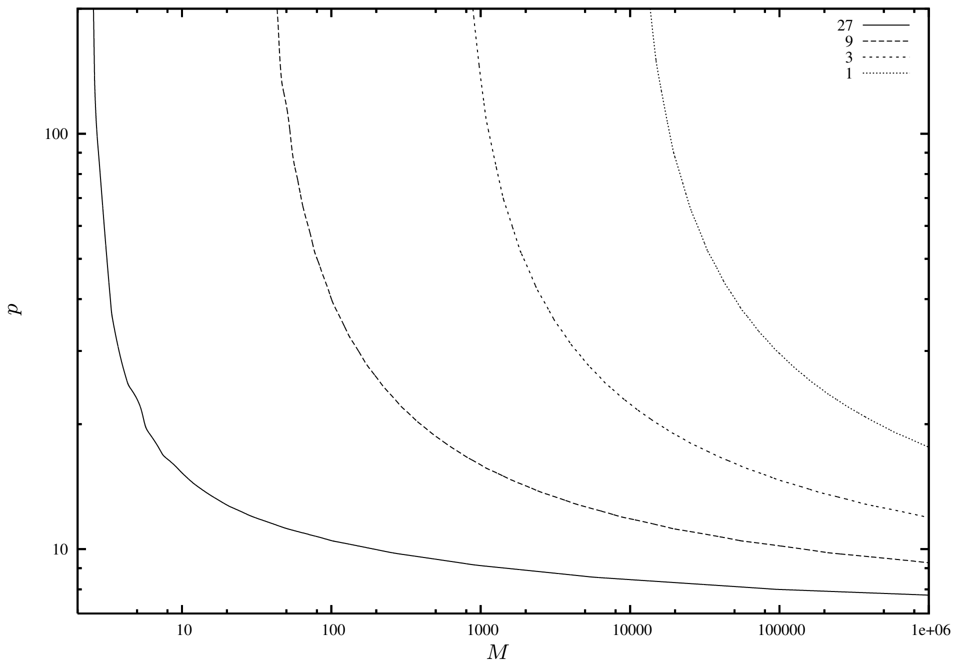

Figure 3: Isolines of σ_0 for large values of p and M

Now we fix $M = 3$ and $p = 20$. Numerically realizing the recipe of the previous subsection yields approximately $\tau_0 = 68.6$ and $\sigma_0 = 34.3$, calculated with roughly the choice $D_1 = .5377$.

We set

$$\tau := \left\lceil \frac{\tau_0}{\sqrt{5}} \right\rceil \sqrt{5} \geq \tau_0(M, p).$$

Then V_τ is p-admissible by the definition of τ_0. Defining $V(x) := 5V_\tau(\sqrt{5}\,x)$ equation (4.2) is equivalent with

$$-v'' + V(x)v = |v|^{p-2}v, \qquad v \in H^1(\mathbb{R}^N),$$

under the scaling

$$v(x) := 5^{\frac{1}{p-2}} u(\sqrt{5}\,x).$$

This new potential V is the one presented in Example 1.5. It has the data $\min V = 5$, $\max V = 15$, and period $\tau/\sqrt{5} = 31$. The rescaling leaves p-admissibility invariant (although it changes c_0), that is, also V is p-admissible.

4.2 Remark. The actual calculation of $\tau_0(M,p)$ and $\sigma_0(M,p)$ for different values of M and p presented here is realized as a program written in the language C, using the GNU compiler *gcc* and the mathematical library *GNU gsl*. For the inversion of the functions g_2 and g_3 we use the root finding algorithm *gsl_root_fdfsolver_steffenson*, and for minimizing τ_1 over D_1 we use the minimizing algorithm *gsl_min_fminimizer_brent* of the gsl library.

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