BOUNDARY CLUSTERED LAYERS NEAR THE HIGHER CRITICAL EXPONENTS

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ABSTRACT. We consider the supercritical problem

$$-\Delta u = |u|^{p-2}\,u \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega,$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ and p smaller than the critical exponent $2^*_{N,k} := \frac{2(N-k)}{N-k-2}$ for the Sobolev embedding of $H^1(\mathbb{R}^{N-k})$ in $L^q(\mathbb{R}^{N-k})$, $1 \leq k \leq N-3$. We show that in some suitable domains Ω there are positive and sign changing solutions with positive and negative layers which concentrate along one or several k-dimensional submanifolds of $\partial\Omega$ as p approaches $2^*_{N,k}$ from below.

KEY WORDS: Nonlinear elliptic boundary value problem; critical and supercritical exponents; existence of positive and sign changing solutions.

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1. INTRODUCTION

Consider the classical Lane-Emden-Fowler problem

$$\Delta v + |v|^{p-2}v = 0 \quad \text{in} \ \ \mathcal{D}, \qquad v = 0 \quad \text{on } \partial\mathcal{D}, \tag{1}$$

where $\mathcal{D}$ is a bounded smooth domain in $\mathbb{R}^N$ and $p > 2$.

It is well known that when p is smaller than the critical Sobolev exponent $2^* := \frac{2N}{N-2}$, compactness of the Sobolev embedding ensures the existence of at least one positive solution and infinitely many sign changing solutions. In contrast, existence of solutions to problem (1) when $p \geq 2^*$ is a delicate issue. Pohozhaev's identity [22] implies that problem (1) does not have a nontrivial solution if the domain $\mathcal{D}$ is strictly starshaped. On the other hand, Kazdan and Warner showed in [13] that if the domain $\mathcal{D}$ is an annulus, problem (1) has infinitely many radial solutions.

For the critical case $p = 2^*$ Bahri and Coron [1] proved that a positive solution of (1) exists if the domain $\mathcal{D}$ has nontrivial reduced homology with $\mathbb{Z}/2$-coefficients. Moreover, it was proved by Ge, Musso and Pistoia [11] and Musso and Pistoia [16] that, if $\mathcal{D}$ has a small hole, problem (1) has many sign changing solutions, whose number increases as the diameter of the hole decreases. Multiplicity results are also available for domains which are not small perturbations of a given domain, but have enough, possibly finite, symmetries, as proved by Clapp and Pacella [8] and Clapp and Faya [6].

The almost critical case $p = 2^* \pm \epsilon$, with ϵ positive and small enough, has been widely studied. The slightly subcritical case $p = 2^* - \epsilon$ was considered by Bahri, Li and Rey [2] and Rey [23], who showed the existence of positive solutions

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which blow-up at one or more points of $\mathcal{D}$ as $\epsilon \to 0$. A large number of sign changing solutions with simple or multiple positive and negative blow-up points were constructed by Bartsch, Micheletti and Pistoia [3], Musso and Pistoia [17], and Pistoia and Weth [21]. For the slightly supercritical case $p = 2^* + \epsilon$ existence and nonexistence of positive solutions with one or more blow-up points has been established by Ben Ayed, El Mehdi, Grossi and Rey [9], Pistoia and Rey [20], and del Pino, Felmer and Musso [5].

Unlike the critical case, in the supercritical case $p > 2^*$ the existence of a nontrivial homology class in $\mathcal{D}$ does not guarantee the existence of a nontrivial solution to (1). In fact, for each integer k such that $1 \leq k \leq N-3$, Passaseo [18, 19] exhibited a bounded smooth domain in $\mathbb{R}^N$, homotopically equivalent to the k-dimensional sphere, in which problem (1) does not have a nontrivial solution for $p \geq 2^*_{N,k} := \frac{2(N-k)}{N-k-2}$. Note that $2^*_{N,k}$ is the critical Sobolev exponent in dimension $N-k$. Examples of domains with richer homology were recently given by Clapp, Faya and Pistoia [7], where it was shown that for $p > 2^*_{N,k}$ there are bounded smooth domains in $\mathbb{R}^N$ whose cup-length is $k+1$, in which problem (1) does not have a nontrivial solution. On the other hand, for $p = 2^*_{N,k}$ existence of infinitely many solutions in some domains has been recently established by Wei and Yan [25]. Further multiplicity results may be found in [7].

In [10] del Pino, Musso and Pacard considered the case $p = 2^*_{N,1} - \epsilon$ and proved that for some suitable domains $\mathcal{D}$, if ϵ is positive, small enough and different from an explicit set of values, problem (1) has a positive solution which concentrates along a 1-dimensional submanifold of the boundary $\partial\mathcal{D}$. In the same paper, the authors ask the question whether one can find solutions which concentrate at a k-dimensional submanifold for p slightly below $2^*_{N,k}$. More precisely, they ask the following:

Problem 1.1. Given $1 \leq k \leq N-3$, are there domains $\mathcal{D}$ in which problem *(1)* has a positive solution v_p for each $p < 2^*_{N,k}$ with the property that these solutions concentrate along a k-dimensional submanifold of the boundary $\partial\mathcal{D}$ as $p \to 2^*_{N,k}$?

Having in mind that when p approaches the first critical exponent 2^* from below a large number of sign changing solutions exist, another question arises naturally:

Problem 1.2. Given $1 \leq k \leq N-3$, are there domains $\mathcal{D}$ in which problem *(1)* has a sign changing solution v_p for each $p < 2^*_{N,k}$ with the property that these solutions concentrate along a k-dimensional submanifold of the boundary $\partial\mathcal{D}$ as $p \to 2^*_{N,k}$?

In this paper, we give a positive answer to both questions. In particular, for each set of positive integers $k_1, \ldots, k_m$ with $k := k_1 + \cdots + k_m \leq N-3$ we exhibit domains $\mathcal{D}$ in which problem (1) has a positive solution for each $p < 2^*_{N,k}$ and, as $p \to 2^*_{N,k}$, these solutions concentrate along a k-dimensional submanifold M of the boundary $\partial\mathcal{D}$ which is diffeomorphic to the product of spheres $\mathbb{S}^{k_1} \times \cdots \times \mathbb{S}^{k_m}$. Moreover, problem (1) has also a sign changing solution with a positive and a negative layer, both of which concentrate along M as $p \to 2^*_{N,k}$. This follows from our main results, which we next state.

Fix $k_1, \ldots, k_m \in \mathbb{N}$ with $k := k_1 + \cdots + k_m \leq N-3$ and a bounded smooth domain Ω in $\mathbb{R}^{N-k}$ such that

$$\overline{\Omega} \subset \{(x_1, \ldots, x_m, x') \in \mathbb{R}^m \times \mathbb{R}^{N-k-m} : x_i > 0,\ i = 1, \ldots, m\}. \tag{2}$$

Set

(3)
$$\mathcal{D} := \{(y^1,\dots,y^m,z) \in \mathbb{R}^{k_1+1}\times\cdots\times\mathbb{R}^{k_m+1}\times\mathbb{R}^{N-k-m} : \left(\left|y^1\right|,\dots,\left|y^m\right|,z\right)\in\Omega\}.$$

$\mathcal{D}$ is a bounded smooth domain in $\mathbb{R}^N$ which is invariant under the action of the group $\Gamma := O(k_1+1)\times\cdots\times O(k_m+1)$ on $\mathbb{R}^N$ given by

$$(g_1,\dots,g_m)(y^1,\dots,y^m,z) := (g_1y^1,\dots,g_my^m,z).$$

for every $g_i \in O(k_i+1)$, $y^i \in \mathbb{R}^{k_i+1}$, $z \in \mathbb{R}^{N-k-m}$. Here, as usual, $O(d)$ denotes the group of all linear isometries of $\mathbb{R}^d$. For $p = 2^*_{N,k} - \epsilon$ we shall look for Γ-invariant solutions to problem (1), i.e. solutions v of the form

$$v(y^1,\dots,y^m,z) = u(\left|y^1\right|,\dots,\left|y^m\right|,z). \tag{4}$$

A simple calculation shows that v solves problem (1) if and only if u solves

$$-\Delta u - \sum_{i=1}^{m}\frac{k_i}{x_i}\frac{\partial u}{\partial x_i} = |u|^{p-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

This problem can be rewritten as

$$-\mathrm{div}(a(x)\nabla u) = a(x)|u|^{p-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $a(x_1,\dots,x_{N-k}) := x_1^{k_1}\cdots x_m^{k_m}$. Note that $2^*_{N,k}$ is the critical exponent in dimension $n := N-k$ which is the dimension of Ω.

Thus, we are lead to study the more general almost critical problem

$$-\mathrm{div}(a(x)\nabla u) = a(x)\,|u|^{\frac{4}{n-2}-\epsilon}\,u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{5}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, $n \geq 3$, ϵ is a positive parameter, and $a \in \mathcal{C}^2(\overline{\Omega})$ is strictly positive on $\overline{\Omega}$.

This is a subcritical problem, so standard variational methods yield one positive and infinitely many sign changing solutions to problem (5) for every $\epsilon \in (0, \frac{4}{n-2})$, cf. Proposition 4.1 in [7]. Our goal is to construct solutions u_ϵ with positive and negative bubbles which accumulate at some points $\xi_1,\dots,\xi_\kappa$ of $\partial\Omega$ as $\epsilon \to 0$. They correspond, via (4), to Γ-invariant solutions v_ϵ of problem (1) with positive and negative layers which accumulate along the k-dimensional submanifolds

$$M_j := \{(y^1,\dots,y^m,z) \in \mathbb{R}^{k_1+1}\times\cdots\times\mathbb{R}^{k_m+1}\times\mathbb{R}^{N-k-m} : \left(\left|y^1\right|,\dots,\left|y^m\right|,z\right) = \xi_j\}$$

of the boundary of $\mathcal{D}$ as $\epsilon \to 0$. Note that each M_j is diffeomorphic to $\mathbb{S}^{k_1}\times\cdots\times\mathbb{S}^{k_m}$ where $\mathbb{S}^d$ is the unit sphere in $\mathbb{R}^{d+1}$.

We will assume one of the following conditions.

($a1$) There exist κ nondegenerate critical points $\xi_1,\dots,\xi_\kappa \in \partial\Omega$ of the restriction of a to $\partial\Omega$ such that

$$\langle\nabla a(\xi_i),\nu(\xi_i)\rangle > 0 \qquad \forall i = 1,\dots,\kappa,$$

where $\nu(\xi_i)$ is the inward pointing unit normal to $\partial\Omega$ at ξ_i.

($a2$) There exists a critical point $\xi_0 \in \partial\Omega$ of the restriction of a to $\partial\Omega$ such that $\langle\nabla a(\xi_0),\nu(\xi_0)\rangle > 0$, and vectors $\tau_1,\dots,\tau_{n-1} \in \mathbb{R}^n$ such that the set $\{\nu(\xi_0),\tau_1,\dots,\tau_{n-1}\}$ is orthonormal and Ω and a are invariant with respect to the reflection ϱ_i on each of the hyperplanes $\xi_0 + \{\tau_i = 0\}$, i.e.

$$\varrho_i(x) \in \Omega \qquad \text{and} \qquad a(\varrho_i(x)) = a(x) \qquad \forall x \in \Omega,$$

$i = 1, ..., n-1$, where

$$\varrho_i(\xi_0 + \langle x, \nu \rangle \nu + \langle x, \tau_1 \rangle \tau_1 + \cdots + \langle x, \tau_i \rangle \tau_i + \cdots + \langle x, \tau_{n-1} \rangle \tau_{n-1})$$
$$= \xi_0 + \langle x, \nu \rangle \nu + \langle x, \tau_1 \rangle \tau_1 + \cdots - \langle x, \tau_i \rangle \tau_i + \cdots + \langle x, \tau_{n-1} \rangle \tau_{n-1}.$$

and $\nu := \nu(\xi_0)$ is the inward pointing unit normal to $\partial\Omega$ at ξ_0.

For each $\delta > 0,\ \xi \in \mathbb{R}^n$, we consider the standard bubble

$$U_{\delta,\xi}(x) := [n(n-2)]^{\frac{n-2}{4}} \frac{\delta^{\frac{n-2}{2}}}{(\delta^2 + |x-\xi|^2)^{\frac{n-2}{2}}}.$$

We prove the following results.

Theorem 1.3. *Assume that $(a1)$ holds true. Then there exists $\epsilon_0 > 0$ such that, for each $\lambda_1, \ldots, \lambda_\kappa \in \{0,1\}$ and $\epsilon \in (0, \epsilon_0)$, problem (5) has a solution u_ϵ which satisfies*

$$u_\epsilon(x) = \sum_{i=1}^{\kappa} (-1)^{\lambda_i} U_{\delta_{i,\epsilon}, \xi_{i,\epsilon}}(x) + o(1) \qquad \text{in } D^{1,2}(\Omega),$$

with

$$\epsilon^{-\frac{n-1}{n-2}} \delta_{i,\epsilon} \to d_i > 0 \qquad \text{and} \qquad \xi_{i,\epsilon} \to \xi_i \in \partial\Omega,$$

for each $i = 1, \ldots, \kappa$, as $\epsilon \to 0$.

Theorem 1.4. *Assume that $(a2)$ holds true. Then there exists $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0)$, problem (5) has a sign changing solution u_ϵ which is invariant with respect to each reflection ϱ_i, $i = 1, ..., n-1$, and satisfies*

$$u_\epsilon(x) = U_{\delta_{1,\epsilon}, \xi_{1,\epsilon}}(x) - U_{\delta_{2,\epsilon}, \xi_{2,\epsilon}}(x) + o(1) \qquad \text{in } D^{1,2}(\Omega),$$

with

$$\epsilon^{-\frac{n-1}{n-2}} \delta_{i,\epsilon} \to d_i > 0, \quad \xi_{i,\epsilon} = \xi_0 + \epsilon t_{i,\epsilon} \nu(\xi_0) \quad \text{and} \quad t_{i,\epsilon} \to t_i > 0,$$

for each $i = 1, 2$, as $\epsilon \to 0$.

Theorem 1.4 states the existence of a sign changing solution whose two blow-up points (one positive and one negative) collapse to the same point ξ_0 of the boundary of Ω under the symmetry assumption $(a2)$.

Some interesting questions arise:

Problem 1.5. Is it possible to find sign changing solutions with $k \geq 3$ blow-up points with alternating sign which collapse to the point ξ_0?

Problem 1.6. Is it possible to find a sign changing solution with one positive and one negative blow-up point which collapse to the point ξ_0 in the more general case when ξ_0 is a nondegenerate critical point of a constrained to $\partial\Omega$ such that $\langle \nabla a(\xi_0), \nu(\xi_0) \rangle > 0$, without any symmetry assumption?

The reason for including the symmetry assumption $(a2)$ in Theorem 1.4 is that it allows to simplify the computations considerably (see Remark 2.6).

In the following two theorems we assume we are given $k_1, \ldots, k_m \in \mathbb{N}$ with $k := k_1 + \cdots + k_m \leq N-3$ and a bounded smooth domain Ω in $\mathbb{R}^{N-k}$ which satisfies (2). We set $a(x_1, \ldots, x_{N-k}) := x_1^{k_1} \cdots x_m^{k_m}$, $\mathcal{D}$ as in (3), $p = 2^*_{N,k} - \epsilon$, $\Gamma := O(k_1+1) \times \cdots \times O(k_m+1)$ and

$$\widetilde{U}_{\delta,\xi}(y^1, \ldots, y^m, z) := U_{\delta,\xi}(\left|y^1\right|, \ldots, \left|y^m\right|, z)$$

for $\delta > 0,\ \xi \in \mathbb{R}^{N-k}$.

Theorem 1.7. *Assume that* (*a*1) *holds true for* a *and* Ω *as above. Then there exists* $\epsilon_0 > 0$ *such that, for each* $\lambda_1, \ldots, \lambda_\kappa \in \{0,1\}$ *and* $\epsilon \in (0, \epsilon_0)$*, problem* (1) *has a* Γ*-invariant solution* v_ϵ *which satisfies*

$$v_\epsilon(x) = \sum_{i=1}^{\kappa} (-1)^{\lambda_i} \widetilde{U}_{\delta_{i,\epsilon},\xi_{i,\epsilon}}(x) + o(1) \qquad \text{in } D^{1,2}(\mathcal{D}),$$

with

$$\epsilon^{-\frac{n-1}{n-2}} \delta_{i,\epsilon} \to d_i > 0 \qquad \text{and} \qquad \xi_{i,\epsilon} \to \xi_i \in \partial\Omega,$$

for each $i = 1, \ldots, \kappa$*, as* $\epsilon \to 0$.

Theorem 1.8. *Assume that* (*a*2) *holds true for* a *and* Ω *as above. Then there exists* $\epsilon_0 > 0$ *such that, for each* $\epsilon \in (0, \epsilon_0)$*, problem* (1) *has a* Γ*-invariant sign changing solution* v_ϵ *which satisfies*

$$v_\epsilon(x) = \widetilde{U}_{\delta_{1,\epsilon},\xi_{1,\epsilon}}(x) - \widetilde{U}_{\delta_{2,\epsilon},\xi_{2,\epsilon}}(x) + o(1) \qquad \text{in } D^{1,2}(\mathcal{D}),$$

with

$$\epsilon^{-\frac{n-1}{n-2}} \delta_{i,\epsilon} \to d_i > 0, \quad \xi_{i,\epsilon} = \xi_0 + \epsilon t_{i,\epsilon} \nu(\xi_0) \quad \text{and} \quad t_{i,\epsilon} \to t_i > 0,$$

for each $i = 1, 2,$ *as* $\epsilon \to 0$.

By the previous discussion Theorems 1.7 and 1.8 follow immediately from Theorems 1.3 and 1.4. The proof of Theorems 1.3 and 1.4 relies on a very well known Ljapunov-Schmidt reduction procedure. We shall omit many details on this procedure because they can be found, up to some minor modifications, in the literature. We only compute what cannot be deduced from known results.

The outline of the paper is as follows: In Section 2 we write the approximate solution, sketch the Ljapunov-Schmidt procedure and use it to prove Theorems 1.3 and 1.4. In Appendix B we compute the rate of the error term and in Appendix C we estimate the reduced energy. In Appendix A we give some important estimates on the Green function close to the boundary.

2. The variational setting

We take

$$(u, v) := \int_\Omega a(x) \nabla u \cdot \nabla v \, dx, \qquad \|u\| := \left(\int_\Omega a(x) \left| \nabla u \right|^2 dx \right)^{1/2},$$

as the inner product in $\mathrm{H}_0^1(\Omega)$ and its corresponding norm. Since a is strictly positive and bounded in $\overline{\Omega}$ they are well defined and equivalent to the standard ones. Similarly, for each $r \in [1, \infty)$,

$$\|u\|_r := \left(\int_\Omega a(x) \left| u \right|^r dx \right)^{1/r}$$

is a norm in $\mathrm{L}^r(\Omega)$ which is equivalent to the standard one.

Next, we rewrite problem (5) in a different way. Let $i^* : \mathrm{L}^{\frac{2n}{n+2}}(\Omega) \to \mathrm{H}_0^1(\Omega)$ be the adjoint operator to the embedding $i : \mathrm{H}_0^1(\Omega) \hookrightarrow \mathrm{L}^{\frac{2n}{n-2}}(\Omega)$, i.e. $i^*(u) = v$ if and only if

$$(v, \varphi) = \int_\Omega a(x) u(x) \varphi(x) dx \quad \text{for all } \varphi \in C_c^\infty(\Omega)$$

if and only if

$$-\operatorname{div}(a(x) \nabla v) = a(x) u \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega.$$

Clearly, there exists a positive constant c such that

$$\|i^*(u)\| \leq c\,\|u\|_{\frac{2n}{n+2}} \qquad \forall\ u \in \mathrm{L}^{\frac{2n}{n+2}}(\Omega).$$

Setting $p := \frac{2n}{n-2}$ and $f_\epsilon(s) := |s|^{p-2-\epsilon}\,s$, problem (5) turns out to be equivalent to

$$u = i^*\left(f_\epsilon(u)\right), \qquad u \in \mathrm{H}^1_0(\Omega). \tag{6}$$

Set $f(s) := f_0(s)$ and $\alpha_n := [n(n-2)]^{\frac{n-2}{4}}$. Let

$$U_{\delta,\xi} := \alpha_n \frac{\delta^{\frac{n-2}{2}}}{(\delta^2+|x-\xi|^2)^{\frac{n-2}{2}}}, \qquad \delta>0, \quad \xi\in\mathbb{R}^n,$$

be the positive solutions to the limit problem

$$-\Delta u = f(u), \qquad u\in H^1(\mathbb{R}^n).$$

Set

$$\psi^0_{\delta,\xi}(x) := \frac{\partial U_{\delta,\xi}}{\partial\delta} = \alpha_n\frac{n-2}{2}\delta^{\frac{n-4}{2}}\frac{|x-\xi|^2-\delta^2}{(\delta^2+|x-\xi|^2)^{n/2}}$$

and, for each $j=1,\dots,n$,

$$\psi^j_{\delta,\xi}(x) := \frac{\partial U_{\delta,\xi}}{\partial\xi_j} = \alpha_n(n-2)\delta^{\frac{n-2}{2}}\frac{x_j-\xi_j}{(\delta^2+|x-\xi|^2)^{n/2}}.$$

Recall that the space spanned by the $(n+1)$ functions $\psi^j_{\delta,\xi}$ is the set of solutions to the linearized problem

$$-\Delta\psi = (p-1)U^{p-2}_{\delta,\xi}\psi \qquad \text{in } \mathbb{R}^n.$$

Let PW denote the projection of the function $W\in D^{1,2}(\mathbb{R}^n)$ onto $\mathrm{H}^1_0(\Omega)$, i.e.

$$\Delta PW = \Delta W \ \text{ in } \Omega, \qquad PW=0 \ \text{ on } \partial\Omega.$$

We look for two different types of solutions to problem (5). The solutions found in Theorem 1.3 are of the form

$$u_\epsilon = \sum_{i=1}^{\kappa}(-1)^{\lambda_i}PU_{\delta_{i,\epsilon},\xi_{i,\epsilon}} + \phi, \tag{7}$$

for fixed $\lambda_i\in\{0,1\}$, where the concentration parameters satisfy

$$\delta_{i,\epsilon} = \epsilon^{\frac{n-1}{n-2}}d_i \quad \text{for some} \quad d_i>0, \tag{8}$$

and the concentration points satisfy

$$\xi_{i,\epsilon} = s_i + \eta_i\nu(s_i) \ \text{ where } s_i\in\partial\Omega \text{ and } \eta_i=\epsilon t_i \text{ for some } t_i>0. \tag{9}$$

Here and in the following $\nu(s_i)$ denotes the inward unit normal to the boundary $\partial\Omega$ at the point s_i.

On the other hand, the solutions found in Theorem 1.4 are of the form

$$u_\epsilon = \sum_{i=1}^{\ell}(-1)^{i+1}PU_{\delta_{i,\epsilon},\xi_{i,\epsilon}} + \phi, \tag{10}$$

where the concentration parameters satisfy (8), while the concentration points are aligned on the line $\mathfrak{L} := \{\xi_0 + r\nu(\xi_0) : r\in\mathbb{R}\}$, namely

$$\xi_{i,\epsilon} = \xi_0 + \eta_i\nu(\xi_0) \qquad \text{where } \eta_i=\epsilon t_i \text{ for some } 0<t_1<\dots<t_\ell. \tag{11}$$

Next, we introduce the configuration space Λ where concentration parameters and concentration points lie. For solutions of type (7) we set $\mathbf{s} = (s_1, \dots, s_\kappa) \in (\partial\Omega)^\kappa$, $\mathbf{d} = (d_1, \dots, d_\kappa) \in (0, +\infty)^\kappa$, and $\mathbf{t} = (t_1, \dots, t_\kappa) \in (0, +\infty)^\kappa$, and so

$$\Lambda := \{(\mathbf{s}, \mathbf{d}, \mathbf{t}) \in (\partial\Omega)^\kappa \times (0, +\infty)^\kappa \times (0, +\infty)^\kappa \ : \ s_i \neq s_j \text{ if } i \neq j\},$$

while for solutions of type (10), we fix $\mathbf{s} = (\xi_0, \dots, \xi_0)$ and we set $\mathbf{d} = (d_1, \dots, d_\ell) \in (0, +\infty)^\ell$, and $\mathbf{t} = (t_1, \dots, t_\ell) \in (0, +\infty)^\ell$, and so

$$\Lambda := \left\{(\mathbf{d}, \mathbf{t}) \in (0, +\infty)^\ell \times (0, +\infty)^\ell \ : \ t_1 < \cdots < t_\ell\right\}.$$

In each of these cases we write

$$V_{\mathbf{s},\mathbf{d},\mathbf{t}} := \sum_{i=1}^{\kappa} (-1)^{\lambda_i} PU_{\delta_i,\xi_i} \qquad \text{and} \qquad V_{\mathbf{s},\mathbf{d},\mathbf{t}} = V_{\mathbf{d},\mathbf{t}} := \sum_{i=1}^{\ell} (-1)^{i+1} PU_{\delta_i,\xi_i}$$

respectively.

The rest term ϕ belongs to a suitable space which we now define. For simplicity we write $\psi_i^j := \psi^j_{\delta_{i,\epsilon},\xi_{i,\epsilon}}$ with $\delta_{i,\epsilon}$ as in (8) and $\xi_{i,\epsilon}$ as in (9) or (11).

For solutions of type (7) we introduce the spaces

$$K_{\mathbf{s},\mathbf{d},\mathbf{t}} := \operatorname{span}\{P\psi_i^j \ : i = 1, \dots, \kappa, \ j = 0, 1, \dots, n\},$$

$$K^\perp_{\mathbf{s},\mathbf{d},\mathbf{t}} := \left\{\phi \in \mathrm{H}^1_0(\Omega) : (\phi, P\psi_i^j) = 0, \ i = 1, \dots, \kappa, \ j = 0, 1, \dots, n\right\}.$$

Note that for $\xi_{i,\epsilon}$ as in (11) the functions $P\psi_i^j$ are invariant with respect to the reflections ϱ_i given in $(a2)$. So for solutions of type (10) we define the space $K_{\mathbf{s},\mathbf{d},\mathbf{t}}$ as above and $K^\perp_{\mathbf{s},\mathbf{d},\mathbf{t}}$ as the orthogonal complement of $K_{\mathbf{s},\mathbf{d},\mathbf{t}}$ in the subspace of all functions in $\mathrm{H}^1_0(\Omega)$ which are invariant with respect to $\varrho_1, \dots, \varrho_{n-1}$. Then we introduce the orthogonal projection operators $\Pi_{\mathbf{s},\mathbf{d},\mathbf{t}}$ and $\Pi^\perp_{\mathbf{s},\mathbf{d},\mathbf{t}}$ in $H^1_0(\Omega)$ with ranges $K_{\mathbf{s},\mathbf{d},\mathbf{t}}$ and $K^\perp_{\mathbf{s},\mathbf{d},\mathbf{t}}$, respectively.

As usual, our approach to solve problem (6) will be to find a $(\mathbf{s}, \mathbf{d}, \mathbf{t}) \in \Lambda$ and a function $\phi \in K^\perp_{\mathbf{s},\mathbf{d},\mathbf{t}}$ such that

$$\Pi^\perp_{\mathbf{s},\mathbf{d},\mathbf{t}} \left(V_{\mathbf{s},\mathbf{d},\mathbf{t}} + \phi - i^* \left[f_\epsilon(V_{\mathbf{s},\mathbf{d},\mathbf{t}} + \phi)\right]\right) = 0 \tag{12}$$

and

$$\Pi_{\mathbf{s},\mathbf{d},\mathbf{t}} \left(V_{\mathbf{s},\mathbf{d},\mathbf{t}} + \phi - i^* \left[f_\epsilon(V_{\mathbf{s},\mathbf{d},\mathbf{t}} + \phi)\right]\right) = 0. \tag{13}$$

First we shall find, for each $(\mathbf{s}, \mathbf{d}, \mathbf{t}) \in \Lambda$ and small ϵ, a function $\phi \in K^\perp_{\mathbf{s},\mathbf{d},\mathbf{t}}$ such that (12) holds. To this aim we define a linear operator $L_{\mathbf{s},\mathbf{d},\mathbf{t}} : K^\perp_{\mathbf{s},\mathbf{d},\mathbf{t}} \to K^\perp_{\mathbf{s},\mathbf{d},\mathbf{t}}$ by

$$L_{\mathbf{s},\mathbf{d},\mathbf{t}}\phi := \phi - \Pi^\perp_{\mathbf{s},\mathbf{d},\mathbf{t}} i^* \left[f'(V_{\mathbf{s},\mathbf{d},\mathbf{t}})\phi\right].$$

The following statement holds true.

Proposition 2.1. *For any compact subset* $\mathbf{C}$ *of* Λ *there exist* $\epsilon_0 > 0$ *and* $c > 0$ *such that for each* $\epsilon \in (0, \epsilon_0)$ *and* $(\mathbf{s}, \mathbf{d}, \mathbf{t}) \in \mathbf{C}$ *the operator* $L_{\mathbf{s},\mathbf{d},\mathbf{t}}$ *is invertible and*

$$\|L_{\mathbf{s},\mathbf{d},\mathbf{t}}\phi\| \geq c \|\phi\| \qquad \forall \ \phi \in K^\perp_{\mathbf{s},\mathbf{d},\mathbf{t}}.$$

Proof. We argue as in Lemma 1.7 of [15]. □

Now we are in position to solve equation (12).

Proposition 2.2. *For any compact subset* $\mathbf{C}$ *of* Λ *there exist* $\epsilon_0, c, \sigma > 0$ *such that for each* $\epsilon \in (0, \epsilon_0)$ *and* $(\mathbf{s}, \mathbf{d}, \mathbf{t}) \in \mathbf{C}$ *there exists a unique* $\phi^\epsilon_{\mathbf{s},\mathbf{d},\mathbf{t}} \in K^\perp_{\mathbf{s},\mathbf{d},\mathbf{t}}$ *such that* (12) *holds and*

$$\left\|\phi^\epsilon_{\mathbf{s},\mathbf{d},\mathbf{t}}\right\| \leq c\epsilon^{\frac{1}{2}+\sigma}. \tag{14}$$

Proof. We estimate the rate of the error term

$$R_{\mathbf{s},\mathbf{d},\mathbf{t}} := \Pi^\perp_{\mathbf{s},\mathbf{d},\mathbf{t}}\left(V_{\mathbf{s},\mathbf{d},\mathbf{t}} - i^*\left[f_\epsilon(V_{\mathbf{s},\mathbf{d},\mathbf{t}})\right]\right) \tag{15}$$

in Appendix B. Then we argue exactly as in Proposition 2.3 of [3]. □

The critical points of the energy functional $J_\epsilon : \mathrm{H}^1_0(\Omega) \to \mathbb{R}$ defined by

$$J_\epsilon(u) := \frac{1}{2}\int_\Omega a(x)|\nabla u|^2 dx - \frac{1}{p-\epsilon}\int_\Omega a(x)|u|^{p-\epsilon} dx$$

are the solutions to problem (5). We define the reduced energy functional $\widetilde{J}_\epsilon : \Lambda \to \mathbb{R}$ by

$$\widetilde{J}_\epsilon(\mathbf{s}, \mathbf{d}, \mathbf{t}) := J_\epsilon(V_{\mathbf{s},\mathbf{d},\mathbf{t}} + \phi^\epsilon_{\mathbf{s},\mathbf{d},\mathbf{t}})$$

The critical points of $\widetilde{J}_\epsilon$ are the solutions to problem (13).

Proposition 2.3. *The function* $V_{\mathbf{s},\mathbf{d},\mathbf{t}} + \phi^\epsilon_{\mathbf{s},\mathbf{d},\mathbf{t}}$ *is a critical point of the functional* J_ϵ *if and only if the point* $(\mathbf{s}, \mathbf{d}, \mathbf{t})$ *is a critical point of the function* $\widetilde{J}_\epsilon$.

Proof. We argue as in Proposition 1 of [2]. □

The problem is thus reduced to the search for critical points of $\widetilde{J}_\epsilon$, so it is necessary to compute the asymptotic expansion of $\widetilde{J}_\epsilon$.

Proposition 2.4. *In case* (7) *it holds true that*

$$\begin{aligned}\widetilde{J}_\epsilon(\mathbf{s}, \mathbf{d}, \mathbf{t}) &= (c_1 + c_2\epsilon\log\epsilon)\sum_{i=1}^{\kappa} a(s_i) \\ &+ \epsilon\sum_{i=1}^{\kappa}\left[c_3 a(s_i) + c_4\langle\nabla a(s_i), \nu(s_i)\rangle t_i + c_5 a(s_i)\left(\frac{d_i}{2t_i}\right)^{n-2} - c_6 a(s_i)\log d_i\right] + o(\epsilon),\end{aligned} \tag{16}$$

C^1*-uniformly on compact sets of* Λ. *Here the* c_i*'s are constants and* c_4, c_5, c_6 *are positive.*

Proof. The proof is postponed to Appendix C. □

Proposition 2.5. *In case* (10) *it holds true that*

$$\widetilde{J}_\epsilon(\mathbf{s}, \mathbf{d}, \mathbf{t}) = \widetilde{J}_\epsilon(\mathbf{d}, \mathbf{t}) = a(\xi_0)\left[c_1 + c_2\epsilon\log\epsilon + c_3\epsilon\right] + \epsilon\Psi(\mathbf{d}, \mathbf{t}) + o(\epsilon), \tag{17}$$

C^0-uniformly on compact sets of Λ. Here

$$\Psi(\mathbf{d},\mathbf{t}) := c_4\langle\nabla a(\xi_0),\nu(\xi_0)\rangle\sum_{i=1}^{\ell} t_i + c_5 a(\xi_0)\times \tag{18}$$

$$\times\left\{\sum_{i=1}^{\ell}\left(\frac{d_i}{2t_i}\right)^{n-2} + \sum_{\substack{i,j=1\\ i\neq j}}^{\ell}(-1)^{i+j+1}(d_i d_j)^{\frac{n-2}{2}}\left[\frac{1}{|t_i-t_j|^{n-2}} - \frac{1}{|t_i+t_j|^{n-2}}\right]\right\}$$

$$- c_6 a(\xi_0)\sum_{i=1}^{\ell}\log d_i$$

where the c_i's are constants and c_4, c_5, c_6 are positive.

Proof. The proof is postponed to Appendix C. □

Proof of Theorem 1.3. Firstly, by Proposition 2.4, we get

$$\widetilde{J}_\epsilon(\mathbf{s},\mathbf{d},\mathbf{t}) = (c_1 + c_2\epsilon\log\epsilon)\sum_{i=1}^{\kappa} a(s_i) + O(\epsilon),$$

C^1-uniformly on compact sets of Λ. Then, since $\xi_1,\ldots,\xi_\kappa$ are non degenerate critical points of a constrained to the boundary of Ω, if ϵ is small enough there exist $\mathbf{s}_\epsilon := (s_{1,\epsilon},\ldots,s_{\kappa,\epsilon})$ such that each $s_{i,\epsilon}\to\xi_i$ as ϵ goes to zero, and $\nabla_{\mathbf{s}}\widetilde{J}_\epsilon(\mathbf{s}_\epsilon,\mathbf{d},\mathbf{t}) = 0$. Secondly, by Proposition 2.4, we also get

$$\begin{aligned}
&\widetilde{J}_\epsilon(\mathbf{s}_\epsilon,\mathbf{d},\mathbf{t}) - (c_1 + c_2\epsilon\log\epsilon)\sum_{i=1}^{\kappa} a(s_{i,\epsilon})\\
&\quad = \epsilon\sum_{i=1}^{\kappa}\Bigg[c_3 a(s_{i,\epsilon}) + c_4\langle\nabla a(s_{i,\epsilon}),\nu(s_{i,\epsilon})\rangle t_i\\
&\qquad\qquad + c_5 a(s_{i,\epsilon})\left(\frac{d_i}{2t_i}\right)^{n-2} - c_6 a(s_{i,\epsilon})\log d_i\Bigg] + o(\epsilon)\\
&\quad = \epsilon\sum_{i=1}^{\kappa}\Bigg[c_3 a(\xi_i) + c_4\langle\nabla a(\xi_i),\nu(\xi_i)\rangle t_i\\
&\qquad\qquad + c_5 a(\xi_i)\left(\frac{d_i}{2t_i}\right)^{n-2} - c_6 a(\xi_i)\log d_i\Bigg] + o(\epsilon).
\end{aligned}$$

It is easy to verify that the function

$$(\mathbf{d},\mathbf{t})\to\sum_{i=1}^{\kappa}\left[c_4\langle\nabla a(\xi_i),\nu(\xi_i)\rangle t_i + c_5 a(\xi_i)\left(\frac{d_i}{2t_i}\right)^{n-2} - c_6 a(\xi_i)\log d_i\right]$$

has a minimum point which is stable under C^0-perturbations. Therefore, there exists a point $(\mathbf{d}_\epsilon,\mathbf{t}_\epsilon)$ such that $\nabla_{(\mathbf{d},\mathbf{t})}\widetilde{J}_\epsilon(\mathbf{s}_\epsilon,\mathbf{d}_\epsilon,\mathbf{t}_\epsilon) = 0$. Thus, the function $\widetilde{J}_\epsilon$ has a critical point and the claim follows from Proposition 2.3. □

Proof of Theorem 1.4. In this case $\ell = 2$ and function Ψ defined in (18) reduces to

$$\begin{aligned}\Psi(\mathbf{d}, \mathbf{t}) &= c_4 \langle a(\xi_0), \nu(\xi_0)\rangle (t_1 + t_2)\\ &+ c_5 a(\xi_0) \left\{ \left(\frac{d_1}{2t_1}\right)^{n-2} + \left(\frac{d_2}{2t_2}\right)^{n-2} + 2(d_1 d_2)^{\frac{n-2}{2}} \left[\frac{1}{|t_i - t_j|^{n-2}} - \frac{1}{|t_i + t_j|^{n-2}} \right] \right\}\\ &- c_6 a(\xi_0)(\log d_1 + \log d_2).\end{aligned}$$

It is easy to verify that it has minimum point which is stable under C^0-perturbations. Therefore, from Proposition 2.5 we deduce that, if ϵ is small enough, the function $\widetilde{J}_\epsilon$ has a critical point. Now the claim follows from Proposition 2.3. □

Remark 2.6. The symmetry assumption $(a2)$ allows to overcome some technical difficulties which arise when looking for a solution whose bubbles collapse to the same point. Indeed, the problem arises when we study the reduced energy and we have to compute the contribution of each peak and the interaction among the peaks. The contribution of each peak is clear: it is given by the distance from the peak to the boundary as in (64) and by the value of the function a at the projection of the peak onto the boundary as in (58). On the other hand, to compute the interaction among the peaks (see (65)) it is important to compare the geodesic distance $d(s_i, s_j)$ between the projections of the peaks onto the boundary with the distance $|\eta_i \nu(s_i) - \eta_j \nu(s_j)|$ between the normal components of the peaks. To have a good expansion the distance $d(s_i, s_j)$ should be negligible with respect to the distance $|\eta_i \nu(s_i) - \eta_j \nu(s_j)|$. But then, in order to find a criticality in the points s_i, we need to go further in the expansion and computations become too tedious. If the domain Ω and the function a are symmetric, we can overcome this difficulty just by assuming that the peaks satisfy (11), so that $d(s_i, s_j) = 0$. In this case the interaction among the peaks is clear and it is given in terms of the Green function of the Laplace operator on the half-space (see (65)).

Appendix A. Boundary estimates of the Green function

In this section we establish the technical estimates we used in the previous part. We denote by $G(x, y)$ the Green function of the Laplacian with Dirichlet boundary condition and by $H(x, y)$ its regular part, i.e.

$$G(x, y) = \frac{1}{n(n-2)\omega_n |x - y|^{n-2}} - H(x, y),$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

First of all, we need an accurate estimate of $H(x, y)$ when the points x and y are close to the boundary. Let us introduce some notation. For $\eta > 0$ we write $\Omega_\eta := \{x \in \Omega \;:\; \mathrm{dist}(x, \partial\Omega) \le \eta\}$. We fix η small enough so that the orthogonal projection $p : \Omega_{2\eta} \to \partial\Omega$ onto the boundary is well defined, i.e. so that for each $x \in \Omega_{2\eta}$ there is a unique point $p(x) \in \partial\Omega$ with $\mathrm{dist}(x, \partial\Omega) = |p(x) - x|$. Set $d_x := \mathrm{dist}(x, \partial\Omega)$, $p_x := p(x)$, and $\nu_x := \nu(x)$, where as before $\nu(x)$ denotes the inward normal to $\partial\Omega$ at x. For $x \in \Omega_{2\eta}$ we define $\bar{x} := p_x - d_x \nu_x = x - 2d_x \nu_x$. Thus, $\bar{x}$ is the reflection of x on $\partial\Omega$.

Lemma A.1. *There exists $C > 0$ such that*

$$\left| H(x,y) - \frac{1}{|\bar{x} - y|^{n-2}} \right| \le \frac{C d_x}{|\bar{x} - y|^{n-2}} \tag{19}$$

$$\left| \nabla_x \left(H(x,y) - \frac{1}{|\bar{x} - y|^{n-2}} \right) \right| \le \frac{C}{|\bar{x} - y|^{n-2}} \tag{20}$$

for all $x \in \Omega_\eta$ and $y \in \Omega$. In particular, there exists $C > 0$ such that

$$0 \le H(x,y) \le \frac{C}{|\bar{x} - y|^{n-2}}, \qquad x \in \Omega_\eta,\ y \in \Omega \tag{21}$$

and

$$|\nabla_x H(x,y)| \le \frac{C}{|x - y|^{n-1}} \qquad x, y \in \Omega. \tag{22}$$

Proof. For convenience we set

$$\chi(x,y) := H(x,y) - \frac{1}{|\bar{x} - y|^{n-2}}$$

for $x \in \Omega_\eta$ and $y \in \Omega$. Note that there is $c > 0$, only dependent on n and η, such that $|\bar{x} - \bar{\xi}| \le c|x - \xi|$ if $x \in \Omega_\eta$ and $\xi \in \overline{B}(x, d_x/2)$. If moreover $y \in \Omega$, then

$$\frac{|\bar{x} - y|}{|\bar{\xi} - y|} \le \frac{|\bar{x} - \bar{\xi}| + |\bar{\xi} - y|}{|\bar{\xi} - y|} \le 1 + \frac{c d_x/2}{|\bar{\xi} - y|} \le 1 + c, \tag{23}$$

since $y \in \Omega$ and $\operatorname{dist}(\bar{\xi}, \Omega) \ge d_x/2$.

The proof of (19) is analogous to the proof of Eq. (2.7) in [4], with obvious small changes. Similarly, slight modifications of the proof of Eq. (2.8) in [4] yield

$$|\Delta_x \chi(x,y)| \le \frac{C}{d_x |\bar{x} - y|^{n-2}} \tag{24}$$

for all $x \in \Omega_\eta$ and $y \in \Omega$. Fix x, y, take $r := 2\sqrt{n}$ and set

$$Q := \{\xi \in \mathbb{R}^n \mid |x - \xi|_\infty \le d_x/r\}.$$

Note that if $\xi \in Q$ then $\xi \in \overline{B}(x, d_x/2)$ and therefore

$$d_x/2 \le d_\xi \le 3d_x/2. \tag{25}$$

Hence we obtain for $i \in \{1, 2, \dots, n\}$

$$\begin{aligned}
|\partial_{x_i} \chi(x,y)| &\le \frac{rn}{d_x} \sup_{\xi \in \partial Q} |\chi(\xi,y)| + \frac{d_x}{2r} \sup_{\xi \in Q} |\Delta_\xi \chi(\xi,y)| && \text{by [12, Eq. (3.15)]} \\
&\le C \left(\sup_{\xi \in \partial Q} \frac{d_\xi}{d_x |\bar{\xi} - y|^{n-2}} + \sup_{\xi \in Q} \frac{d_x}{d_\xi |\bar{\xi} - y|^{n-2}} \right) && \text{by (19) and (24)} \\
&\le C \sup_{\xi \in Q} \frac{1}{|\bar{\xi} - y|^{n-2}} && \text{by (25)} \\
&\le \frac{C}{|\bar{x} - y|^{n-2}} && \text{by (23).}
\end{aligned}$$

Summing up this inequality over i gives (20).

To prove (22), note first that there is $C > 0$ such that

$$|\nabla_x H(x,y)| \le C \qquad \text{if } x \in \Omega \backslash \Omega_\eta,\ y \in \Omega. \tag{26}$$

The case $x \in \Omega_\eta$ relies on the estimate (20). Note that there is $C > 0$ such that

$$\frac{|\bar{x} - y|}{|x - y|} \geq C \qquad \text{for all } x \in \Omega_\eta,\ y \in \Omega. \tag{27}$$

This implies that the term on the right of (20) is estimated by a constant multiple of $1/|x-y|^{n-2}$ if $x \in \Omega_\eta$ and $y \in \Omega$. In view of (26) it therefore remains to show that

$$\left|\nabla_x \frac{1}{|\bar{x} - y|^{n-2}}\right| \leq \frac{C}{|x - y|^{n-1}} \qquad x \in \Omega_\eta,\ y \in \Omega \tag{28}$$

for some constant $C > 0$.

Writing ∂_i for $\partial/\partial x_i$ we calculate as in [4] for any $i \in \{1, 2, \dots, n\}$:

$$\partial_i \frac{1}{|\bar{x} - y|^{n-2}} = \frac{2-n}{|\bar{x} - y|^n} \sum_{j=1}^n (\bar{x}_j - y_j)\partial_i \bar{x}_j. \tag{29}$$

Since $\bar{x} := x - 2d_x\nu_x$, we find

$$\partial_i \bar{x}_j = \delta_{ij} - 2\nu_{xi}\nu_{xj} - 2d_x \partial_i \nu_{xj}.$$

Using this representation in (29) yields

$$\left|\partial_i \frac{1}{|\bar{x} - y|^{n-2}}\right| \leq \frac{C}{|\bar{x} - y|^{n-1}}(1 + d_x|\partial_i \nu_x|).$$

By our choice of η we have $|d_x| \leq \eta$ and $|\partial_i \nu_x| \leq C$ for all $x \in \Omega_\eta$. In view of (27) we obtain (28) and finish the proof. □

Here and in the remaining appendices we employ the notation

$$|u|_{A,q} := \left(\int_A |u|^q\right)^{1/q}$$

for measurable $A \subseteq \mathbb{R}^n$ and $q \in [1, \infty]$. If $A = \Omega$ we omit it from the notation.

Lemma A.2. *Let $\delta, \delta_1, \delta_2 \in (0, 1]$ and $\xi, \xi_1, \xi_2 \in \Omega_\eta$. Let $\bar{\xi}$ be the reflection point of ξ with respect to $\partial\Omega$. There exists $c > 0$ such that*

$$0 \leq PU_{\delta,\xi}(x) \leq U_{\delta,\xi}(x) \tag{30}$$

and

$$0 \leq U_{\delta,\xi}(x) - PU_{\delta,\xi}(x) \leq \alpha_n \delta^{\frac{n-2}{2}} H(x, \xi) \leq c \frac{\delta^{\frac{n-2}{2}}}{|x - \bar{\xi}|^{n-2}} \tag{31}$$

for all $x \in \Omega$. Moreover

$$R_{\delta,\xi}(x) := PU_{\delta,\xi}(x) - U_{\delta,\xi}(x) + \alpha_n \delta^{\frac{n-2}{2}} H(x, \xi)$$

satisfies

$$|R_{\delta,\xi}|_{\Omega,\infty} = O\left(\frac{\delta^{\frac{n+2}{2}}}{\operatorname{dist}(\xi, \partial\Omega)^n}\right). \tag{32}$$

Finally, there is $\beta > 0$ such that

$$\int_\Omega |\nabla PU_{\delta_1,\xi_1}| PU_{\delta_2,\xi_2} = \left(\frac{\delta_1}{\delta_2}\right)^{\frac{n-2}{2}} O\left(\delta_2^{\frac{n-2}{n-1}+\beta}\right) \tag{33}$$

$$|\nabla PU_{\delta,\xi}|_{\frac{2n}{n+2}} = O\big(\delta^{\frac{n-2}{2(n-1)}+\beta}\big) \tag{34}$$

as $\delta, \delta_1, \delta_2 \to 0$, *independently of* ξ, ξ_1, *and* ξ_2.

Proof. Estimates (30), (31), and (32) follow easily from the maximum principle and Lemma A.1.

Note first that

$$|U_{\delta,\xi}|_q = O\left(\delta^{\frac{n}{q} - \frac{n-2}{2}}\right) \qquad \text{if } q > \frac{n}{n-2} \tag{35}$$

and

$$|U_{\delta,\xi}^{\frac{n+2}{n-2}}|_q = O\left(\delta^{\frac{n}{q} - \frac{n+2}{2}}\right) \qquad \text{if } q \geq 1, \tag{36}$$

as $\delta \to 0$, independently of ξ.

Recall that

$$\nabla PU_{\delta,\xi}(x) = \int_\Omega \nabla_x \left(\frac{1}{n(n-2)\omega_n |x-y|^{n-2}} - H(x,y) \right) U_{\delta,\xi}^{\frac{n+2}{n-2}}(y)\, \mathrm{d}y \tag{37}$$

and note that

$$\left| \nabla_x \frac{1}{|x-y|^{n-2}} \right| \leq \frac{C}{|x-y|^{n-1}}. \tag{38}$$

By (37), (22), and (38), to show (33) it suffices to prove

$$\int_\Omega \int_\Omega U_{\delta_2,\xi_2}(x) \frac{1}{|x-y|^{n-1}} U_{\delta_1,\xi_1}^{\frac{n+2}{n-2}}(y)\, \mathrm{d}y\, \mathrm{d}x = \left(\frac{\delta_1}{\delta_2}\right)^{\frac{n-2}{2}} O\left(\delta_2^{\frac{n-2}{n-1}+\beta}\right). \tag{39}$$

For simplicity, set $V := U_{\delta_1,\xi_1}^{\frac{n+2}{n-2}}$ and $g(x) := 1/|x|^{n-1}$. Set $M := \operatorname{diam}(\Omega)$. Pick

$$r \in \left(\frac{n(n-1)}{(n-1)^2+1},\ \frac{n}{n-1} \right)$$

and note that then $r \geq 1$ and $r' > n$, where r' denotes the conjugate exponent of r. Since $\frac{1}{r'} + \frac{1}{r} + 1 = 2$ it follows as in the proof of [14, Theorem 4.2] that

$$\begin{aligned} \int_\Omega \int_\Omega U_{\delta_2,\xi_2}(x) g(x-y) V(y)\, \mathrm{d}y\, \mathrm{d}x &\leq |U_{\delta_2,\xi_2}|_{r'} |g|_{B(0,M),r} |V|_1 \\ &= O\left(\delta_2^{\frac{n}{r'} - \frac{n-2}{2}} \delta_1^{n - \frac{n+2}{2}}\right) = \left(\frac{\delta_1}{\delta_2}\right)^{\frac{n-2}{2}} O\left(\delta_2^{n\left(1-\frac{1}{r}\right)}\right), \end{aligned}$$

by (35) and (36). Here we have used that $|g|_{B(0,M),r}$ is finite since $r < n/(n-1)$. On the other hand, $r > n(n-1)/((n-1)^2+1)$ implies that

$$n\left(1 - \frac{1}{r}\right) = \frac{n-2}{n-1} + \beta$$

for some $\beta > 0$, proving (39) and hence (33).

To prove (34) we proceed similarly. This time we pick

$$s \in \left(\max\left\{1,\ \frac{2n}{n+4}\right\},\ \frac{2n(n-1)}{n^2+2n-4} \right)$$

and define r by

$$\frac{1}{r} + \frac{1}{s} = 1 + \frac{n+2}{2n}. \tag{40}$$

Some basic calculations reveal that s is well defined and that

$$r \in \left[1,\ \frac{n}{n-1}\right). \tag{41}$$

Similarly to the proof of [14, Theorem 4.2], taking into account the Remark (2) following the statement of that theorem, we obtain

$$|\nabla PU_{\delta,\xi}|_{\frac{2n}{n+2}} \le |g|_{B(0,M),r}|V|_s = O\big(\delta^{\frac{n}{s}-\frac{n+2}{2}}\big).$$

Again we have used that $r < n/(n-1)$ implies that the r-norm of g in the ball of radius M is finite. Since $s < 2n(n-1)/(n^2+2n-4)$, there is $\beta > 0$ such that

$$\frac{n}{s} - \frac{n+2}{2} = \frac{n-2}{2(n-1)} + \beta,$$

proving (34). □

Appendix B. An estimate of the error

To simplify notation, from now on we write

$$\delta_i := \delta_{i,\epsilon}, \qquad \xi_i := \xi_{i,\epsilon}, \qquad U_i := U_{\delta_i,\xi_i}.$$

Next, we estimate the error term defined in (15).

Lemma B.1. *It holds true for some $\sigma > 0$ that*

$$\|R_{\mathbf{s},\mathbf{d},\mathbf{t}}\| = O\left(\epsilon^{\frac{1}{2}+\sigma}\right).$$

Proof. We estimate $R_{\mathbf{s},\mathbf{d},\mathbf{t}}$ in case (10). The estimate in case (7) is easier and can be obtained after minor modifications of this argument.

From the definition of i^* we deduce that

$$\begin{aligned}
\|R_{\mathbf{s},\mathbf{d},\mathbf{t}}\| &= O\left(|-\mathrm{div}\,(a(x)\nabla V_{\mathbf{s},\mathbf{d},\mathbf{t}}) - a(x)f_\epsilon\,(V_{\mathbf{s},\mathbf{d},\mathbf{t}})|_{\frac{2n}{n+2}}\right)\\
&= O\left(|-\nabla a\nabla V_{\mathbf{s},\mathbf{d},\mathbf{t}} - a(x)\Delta V_{\mathbf{s},\mathbf{d},\mathbf{t}} - a(x)f_\epsilon\,(V_{\mathbf{s},\mathbf{d},\mathbf{t}})|_{\frac{2n}{n+2}}\right)\\
&= O\left(\sum_i |\nabla a\nabla PU_i|_{\frac{2n}{n+2}}\right) + O\left(\sum_i |a(x)\,[f(U_i) - f(PU_i)]|_{\frac{2n}{n+2}}\right)\\
&\qquad + O\left(|a(x)[\sum_i f(PU_i) - f(\sum_i PU_i)]|_{\frac{2n}{n+2}}\right)\\
&\qquad + O\left(|a(x)\,[f\,(V_{\mathbf{s},\mathbf{d},\mathbf{t}}) - f_\epsilon\,(V_{\mathbf{s},\mathbf{d},\mathbf{t}})]|_{\frac{2n}{n+2}}\right)\\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{42}$$

To estimate I_1 recall that $\delta_i = O\left(\epsilon^{\frac{n-1}{n-2}}\right)$ on compact subsets of Λ. By (34) we get, for some $\sigma > 0$,

$$|\nabla a\nabla PU_i|_{\frac{2n}{n+2}} = O\left(\epsilon^{\frac{1}{2}+\sigma}\right). \tag{43}$$

Let us estimate I_2. By (31) for some $\sigma > 0$ we obtain

$$\begin{aligned}
&|a\,[f(U_i) - f(PU_i)]|_{\frac{2n}{n+2}}\\
&\quad = O\left(|U_i^{p-2}(PU_i - U_i)|_{\frac{2n}{n+2}}\right) + O\left(||PU_i - U_i|^{p-1}|_{\frac{2n}{n+2}}\right) = O\left(\epsilon^{\frac{1}{2}+\sigma}\right),
\end{aligned} \tag{44}$$

because by (31) (using also (48) with $q = (n+2)/4$)

$$
\begin{aligned}
(45) \quad ||PU_i - U_i|^{p-1}|_{\frac{2n}{n+2}} &= |PU_i - U_i|^{p-1}_{\frac{2n}{n-2}} \\
&= \delta_i^{\frac{n+2}{2}} O\left(\left| \frac{1}{|x-\bar{\xi}_i|^{n-2}} \right|^{p-1}_{\frac{2n}{n-2}} \right) = O\left(\delta_i^{\frac{n+2}{2}} \epsilon^{-\frac{n+2}{2}} \right)
\end{aligned}
$$

and by Hölder's inequality for some $\sigma > 0$ (using also (47) and (48) with $q \sim 1$ when $n \le 6$ or $q \sim (n+2)/8$ when $n \ge 7$)

$$
\begin{aligned}
|U_i^{p-2}(PU_i - U_i)|_{\frac{2n}{n+2}} &= \delta_i^{\frac{n-2}{2}} O\left(|U_i|^{p-2}_{\frac{8nq}{(n-2)(n+2)}} \right) O\left(\left| \frac{1}{|x-\bar{\xi}_i|^{n-2}} \right|_{\frac{2nq}{(q-1)(n+2)}} \right) \\
(46) \quad &= \begin{cases} O\left(\left(\dfrac{\delta_i}{\epsilon} \right)^{\frac{n+2}{2}-\sigma} \right) & \text{if } n \ge 7 \\ O\left(\left(\dfrac{\delta_i}{\epsilon} \right)^{n-2-\sigma} \right) & \text{if } n \le 6, \end{cases}
\end{aligned}
$$

with

$$
(47) \qquad |U_i|^{p-2}_{\frac{8nq}{(n-2)(n+2)}} = \begin{cases} O\left(\delta_i^2\right) & \text{if } n \ge 7 \text{ and } 1 < q < \frac{n+2}{8}, \\ O\left(\delta_i^{\frac{n+2}{2q}-2} \right) & \text{if } n \le 6 \text{ and } q > \frac{n+2}{8}. \end{cases}
$$

and

$$
(48) \quad \left| \frac{1}{|x-\bar{\xi}_i|^{n-2}} \right|_{\frac{2nq}{(q-1)(n+2)}} = O\left(\epsilon^{-\frac{n-6}{2}-\frac{n+2}{2q}} \right)
$$

$$
\text{if } n \ge 6 \text{ and } q > 1 \text{ or } n \le 5 \text{ and } 1 < q < \frac{n+2}{6-n}.
$$

Let us estimate I_3. We set

$$
(49) \qquad \eta := \min\left\{ \mathrm{d}(\xi_1, \partial\Omega), \mathrm{d}(\xi_2, \partial\Omega), \frac{|\xi_1 - \xi_2|}{2} \right\}.
$$

We have

$$
\begin{aligned}
(50) \quad & \left| a(x) \left[\sum_i f(PU_i) - f\left(\sum_i PU_i \right) \right] \right|_{\frac{2n}{n+2}} \\
&= O\left(\left| \sum_i f(PU_i) - f\left(\sum_i PU_i \right) \right|_{\Omega \setminus \cup_i B(\xi_i, \eta), \frac{2n}{n+2}} \right) \\
&\quad + O\left(\sum_i \left| f(PU_i) - f\left(\sum_i PU_i \right) \right|_{B(\xi_i, \eta), \frac{2n}{n+2}} \right) \\
&\quad + O\left(\sum_i \sum_{j \ne i} |f(PU_j)|_{B(\xi_i, \eta), \frac{2n}{n+2}} \right),
\end{aligned}
$$

because

$$\left|\sum_i f(PU_i) - f\left(\sum_i PU_i\right)\right|_{\Omega\setminus\cup_i B(\xi_i,\eta),\frac{2n}{n+2}} = O\left(\sum_i |U_i|^{p-1}_{\Omega\setminus B(\xi_i,\eta),\frac{2n}{n-2}}\right) = O\left(\sum_i \left(\frac{\delta_i}{\eta}\right)^n\right) \tag{51}$$

and if $j \neq i$

$$|f(PU_j)|_{B(\xi_i,\eta),\frac{2n}{n+2}} = |U_j|^{p-1}_{B(\xi_i,\eta),\frac{2n}{n-2}} = O\left(|U_j|^{p-1}_{\Omega\setminus B(\xi_j,\eta),\frac{2n}{n-2}}\right) = O\left(\left(\frac{\delta_j}{\eta}\right)^{\frac{n+2}{2}}\right). \tag{52}$$

Moreover

$$\begin{aligned}
&\left|f(PU_i) - f\left(\sum_i PU_i\right)\right|_{B(\xi_i,\eta),\frac{2n}{n+2}} \\
&= O\left(|U_i^{p-2}(PU_i - U_i)|_{B(\xi_i,\eta),\frac{2n}{n+2}}\right) + O\left(\sum_{j\neq i}|U_i^{p-2}U_j|_{B(\xi_i,\eta),\frac{2n}{n+2}}\right) \\
&+ O\left(||PU_i - U_i|^{p-1}|_{B(\xi_i,\eta),\frac{2n}{n+2}}\right) + O\left(\sum_{j\neq i}|U_j^{p-1}|_{B(\xi_i,\eta),\frac{2n}{n+2}}\right)
\end{aligned}$$

and the first term is estimated in (46), the third term is estimated in (45), the fourth term is estimated in (52). The second term is estimated using (47) and (48) (with $q \sim 1$ when $n \leq 6$ or $q \sim (n+2)/8$ when $n \geq 7$) as follows

$$\begin{aligned}
&|U_i^{p-2}U_j|_{B(\xi_i,\eta),\frac{2n}{n+2}} \\
&= \delta_i^{\frac{n-2}{2}} O\left(|U_i|^{p-2}_{B(\xi_i,\eta),\frac{8nq}{(n-2)(n+2)}}\right) O\left(\left|\frac{1}{|x-\xi_j|^{n-2}}\right|_{B(\xi_i,\eta),\frac{2nq}{(q-1)(n+2)}}\right) \\
&= \begin{cases} O\left(\left(\frac{\delta_i}{\epsilon}\right)^{\frac{n+2}{2}-\sigma}\right) & \text{if } n \geq 7 \\ O\left(\left(\frac{\delta_i}{\epsilon}\right)^{n-2-\sigma}\right) & \text{if } n \leq 6, \end{cases}
\end{aligned} \tag{53}$$

for some $\sigma > 0$.

Arguing exactly as in Proposition 2 of [24], we can estimate the last term I_4 by

$$\left|a(x)\left[f\left(V_{\mathbf{s},\mathbf{d},\mathbf{t}}\right) - f_\epsilon\left(V_{\mathbf{s},\mathbf{d},\mathbf{t}}\right)\right]\right|_{\frac{2n}{n+2}} = O\left(\epsilon|\ln\epsilon|\right). \tag{54}$$

□

Appendix C. An estimate of the energy

It is standard to prove that

$$\widetilde{J}_\epsilon(\mathbf{s},\mathbf{d},\mathbf{t}) = J_\epsilon\left(V_{\mathbf{s},\mathbf{d},\mathbf{t}}\right) + \text{h.o.t.}$$

(see for example [3] or [2]), so the problem reduces to estimating the leading term $J_\epsilon\left(V_{\mathbf{s},\mathbf{d},\mathbf{t}}\right)$. We will estimate the leading term in case (10), because the expansion of the leading term in case (7) is easier and can be deduced from that. We also assume $\ell = 2$, because with some minor modifications we treat the general case. Therefore, the estimate will be a direct consequence of Lemma (C.3) and Lemma (C.4).

For future reference we define the constants

$$\gamma_1 = \alpha_n^p \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^n} dy, \tag{55}$$

$$\gamma_2 = \alpha_n^p \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^{\frac{n+2}{2}}} dy, \tag{56}$$

$$\gamma_3 = \alpha_n^p \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^n} \log \frac{1}{(1+|y|^2)^{\frac{n-2}{2}}} dy. \tag{57}$$

We start with the following key estimates.

Lemma C.1. *The following estimate holds true:*

$$\int_{B(\xi_1,\eta)} a(x) U_1^p dx = \gamma_1 a(s_1) + \langle \nabla a(s_1), \nu(s_1)\rangle \gamma_1 t_1 \epsilon + O\left(\epsilon^2\right). \tag{58}$$

Here η is choosen as in (49).

Proof. We split the left-hand side as

$$\int_{B(\xi_1,\eta)} a(x) U_1^p dx = \int_{B(\xi_1,\eta)} a(s_1) U_1^p dx + \int_{B(\xi_1,\eta)} \left(a(x) - a(\xi_1)\right) U_1^p dx. \tag{59}$$

We deduce

$$\int_{B(\xi_1,\eta)} a(\xi_1) U_1^p dx = \gamma_1 a(\xi_0) + O\left(\frac{\delta_1^n}{\eta^n}\right). \tag{60}$$

By the mean value theorem we get

$$\begin{aligned} a(\delta_1 y + \xi_1) - a(\xi_1) &= a(\delta_1 y + \eta_1 \nu(s_1) + s_1) - a(\xi_0) \\ &= \langle \nabla a(s_1), \nu(s_1)\rangle \eta_1 + \delta_1 \langle \nabla a(s_1), y\rangle + R(y), \end{aligned} \tag{61}$$

where R satisfies the uniform estimate

$$|R(y)| \le c\left(\delta_1^2 |y|^2 + \delta_1 \eta_1 |y| + \eta_1^2\right) \text{ for any } y \in B(0, \eta/\delta_1). \tag{62}$$

Therefore we conclude

$$\begin{aligned} &\int_{B(\xi_1,\eta)} \left(a(x) - a(\xi_1)\right) U_1^p dx \\ &= \alpha_n^p \int_{B(0,\eta/\delta_1)} \left[a(\delta y + \eta_1 \nu(s_1) + s_1) - a(s_1)\right] \frac{1}{(1+|y|^2)^n} dy \\ &= \alpha_n^p \int_{B(0,\eta/\delta_1)} \left[\langle \nabla a(s_1), \nu(s_1)\rangle \eta_1 + \delta_1 \langle \nabla a(s_1), y\rangle + R(y)\right] \frac{1}{(1+|y|^2)^n} dy \\ &= \langle \nabla a(s_1), \nu(s_1)\rangle \gamma_1 \eta_1 + O(\eta_1^2). \end{aligned} \tag{63}$$

□

Lemma C.2. *The following estimates hold true:*

$$\int\limits_{B(\xi_1,\eta)} a(x)U_1^{p-1}\left(PU_1-U_1\right)dx=-\gamma_2 a(s_1)\epsilon\left(\frac{d_1}{2t_1}\right)^{n-2}+O\left(\epsilon^{1+\sigma}\right) \tag{64}$$

and

$$\int\limits_{B(\xi_1,\eta)} a(x)U_1^{p-1}PU_2 dx \tag{65}$$
$$=\begin{cases} O\left(\epsilon^{1+\sigma}\right) & if s_1\neq s_2,\\ \gamma_2 a(\xi_0)\epsilon\left(d_1d_2\right)^{\frac{n-2}{2}}\times & \\ \quad\times\left(\frac{1}{|t_1-t_2|^{n-2}}-\frac{1}{|t_1+t_2|^{n-2}}\right)+O\left(\epsilon^{1+\sigma}\right) & if\ s_1=s_2=\xi_0,\end{cases}$$

for some $\sigma>0$. Here η is choosen as in (49).

Proof. First we prove (64). By Lemma A.1 and Lemma A.2 we get

$$\begin{aligned}
&\int\limits_{B(\xi_1,\eta)} a(x)U_1^{p-1}\left(PU_1-U_1\right)dx \\
&=\int\limits_{B(\xi_1,\eta)} a(x)U_1^{p-1}\left(-\alpha_n\delta_1^{\frac{n-2}{2}}H(x,\xi_1)+R_{\delta_1,\xi_1}\right)dx\\
&=-\alpha_n^p\delta_1^{n-2}\int\limits_{B(0,\eta/\delta_1)} a(\delta_1y+\xi_1)H(\delta_1y+\xi_1,\xi_1)\frac{1}{\left(1+|y|^2\right)^{\frac{n+2}{2}}}dy\\
&\qquad+O\left(\left(\frac{\delta_1}{\eta_1}\right)^n\right)\\
&=-\alpha_n^p\delta_1^{n-2}\int\limits_{B(0,\eta/\delta_1)} a(\delta_1y+\xi_1)\frac{1}{|\delta_1y+\xi_1-\bar\xi_1|^{n-2}}\frac{1}{\left(1+|y|^2\right)^{\frac{n+2}{2}}}dy\\
&\qquad+O\left(\left(\frac{\delta_1}{\eta_1}\right)^{n-2}\eta_1\right)+O\left(\left(\frac{\delta_1}{\eta_1}\right)^n\right)\\
&=-\alpha_n^p\left(\frac{\delta_1}{2\eta_1}\right)^{n-2}a(s_1)\int\limits_{\mathbb{R}^n}\frac{1}{\left(1+|y|^2\right)^{\frac{n+2}{2}}}dy\\
&\qquad+O\left(\left(\frac{\delta_1}{\eta_1}\right)^{n-1}\right)+O\left(\left(\frac{\delta_1}{\eta_1}\right)^{n-2}\eta_1\right),
\end{aligned}\tag{66}$$

because

$$|\delta_1y+\xi_1-\bar\xi_1|=|\delta_1y+2\eta_1\nu(s_1)|\geq 2\eta_1-|\delta_1y|\geq\eta_1 \text{ for any } y\in B(0,\eta/\delta_1).$$

and by mean value theorem

$$a(\delta_1y+\xi_1)=a(s_1)+O\left(\eta_1\right) \text{ and } \frac{1}{|\delta_1y+\xi_1-\bar\xi_1|^{n-2}}=\frac{1}{(2\eta_1)^{n-2}}+O\left(\frac{\delta_1|y|}{\eta_1^{n-1}}\right).$$

Next, we prove (65). By Lemma A.2

$$
\begin{aligned}
(67)\quad & \int\limits_{B(\xi_1,\eta)} a(x)U_1^{p-1}PU_2dx \\
&= \int\limits_{B(\xi_1,\eta)} a(x)U_1^{p-1}\left(U_2-\alpha_n\delta_2^{\frac{n-2}{2}}H(x,\xi_2)+R_{\delta_2,\xi_2}\right)dx \\
&= \alpha_n^p(\delta_1\delta_2)^{\frac{n-2}{2}}\int\limits_{B(0,\eta/\delta_1)} a(\delta_1y+\xi_1)\frac{1}{(1+|y|^2)^{\frac{n+2}{2}}}\times \\
&\quad\times\left(\frac{1}{(\delta_2^2+|\delta_1y+\xi_1-\xi_2|^2)^{\frac{n-2}{2}}}dy-H(\delta_1y+\xi_1,\xi_2)\right)dy \\
&\quad+O\left((\delta_1\delta_2)^{\frac{n-2}{2}}\frac{\delta_2^{\frac{n+2}{2}}}{\eta_2^n}\right) \\
&= \alpha_n^p(\delta_1\delta_2)^{\frac{n-2}{2}}\int\limits_{B(0,\eta/\delta_1)} a(\delta_1y+\xi_1)\frac{1}{(1+|y|^2)^{\frac{n+2}{2}}}\times \\
&\quad\times\left(\frac{1}{(\delta_2^2+|\delta_1y+\xi_1-\xi_2|^2)^{\frac{n-2}{2}}}-\frac{1}{|\delta_1y+\xi_1-\bar\xi_2|^{n-2}}\right)dy \\
&\quad+O\left(\frac{(\delta_1\delta_2)^{\frac{n-2}{2}}}{\eta^{n-2}}\eta_2\right)+O\left((\delta_1\delta_2)^{\frac{n-2}{2}}\frac{\delta_2^{\frac{n+2}{2}}}{\eta_2^n}\right) \\
&= \alpha_n^p(\delta_1\delta_2)^{\frac{n-2}{2}}a(\xi_0)\left(\frac{1}{|\eta_1-\eta_2|^{n-2}}-\frac{1}{|\eta_1+\eta_2|^{n-2}}\right)\int\limits_{\mathbb{R}^n}\frac{1}{(1+|y|^2)^{\frac{n+2}{2}}}dy \\
&\quad+O\left(\frac{(\delta_1\delta_2)^{\frac{n-2}{2}}}{\eta^{n-1}}\right)+O\left(\frac{(\delta_1\delta_2)^{\frac{n-2}{2}}}{\eta^{n-1}}\delta_1\right)+O\left((\delta_1\delta_2)^{\frac{n-2}{2}}\frac{\delta_2^{\frac{n+2}{2}}}{\eta_2^n}\right),
\end{aligned}
$$

because for any $y\in B(0,\eta/\delta_1)$ we have

$$|\delta_1y+\xi_1-\bar\xi_2|=|\delta_1y+(\eta_1+\eta_2)\nu(\xi_0)|\geq\eta_1+\eta_2-|\delta_1y|\geq\eta$$

$$|\delta_1y+\xi_1-\xi_2|\geq|\xi_1-\xi_2|-|\delta_1y|\geq\eta$$

and by mean value theorem $a(\delta_1y+\xi_1)=a(\xi_0)+O(\eta_1)$ and

$$
\begin{aligned}
&\frac{1}{(\delta_2^2+|\delta_1y+\xi_1-\xi_2|^2)^{\frac{n-2}{2}}}-\frac{1}{|\delta_1y+\xi_1-\bar\xi_2|^{n-2}} \\
&=\frac{1}{|\eta_1-\eta_2|^{n-2}}-\frac{1}{|\eta_1+\eta_2|^{n-2}}+O\left(\frac{\delta_1|y|+\delta_2^2}{\eta^{n-1}}\right).
\end{aligned}
$$

□

Lemma C.3. *The following estimate holds true:*

$$J_0\left(V_{\mathbf{s},\mathbf{d},\mathbf{t}}\right)=\frac{2-p}{2p}\left[2\gamma_1 a(\xi_0)+\gamma_1\left\langle\nabla a(\xi_0),\nu(\xi_0)\right\rangle\epsilon(t_1+t_2)\right]+\frac{1}{2}\gamma_2 a(\xi_0)\times$$
$$\times\left[\left(\frac{d_1}{2t_1}\right)^{n-2}+\left(\frac{d_2}{2t_2}\right)^{n-2}+2\left(d_1d_2\right)^{\frac{n-2}{2}}\left(\frac{1}{|t_1-t_2|^{n-2}}-\frac{1}{|t_1+t_2|^{n-2}}\right)\right]\epsilon$$
$$+O\left(\epsilon^{1+\sigma}\right),$$

for some $\sigma>0$.

Proof.

$$J_0\left(V_{\mathbf{s},\mathbf{d},\mathbf{t}}\right)=\frac{1}{2}\int_\Omega a(x)|\nabla V_{\mathbf{s},\mathbf{d},\mathbf{t}}|^2dx-\frac{1}{p}\int_\Omega a(x)|V_{\mathbf{s},\mathbf{d},\mathbf{t}}|^p dx \tag{68}$$

We estimate the first term at the R.H.S. of (68). We write

$$\int_\Omega a(x)|\nabla V_{\mathbf{d},\mathbf{t}}|^2dx \tag{69}$$
$$=\int_\Omega a(x)|\nabla PU_1|^2dx+\int_\Omega a(x)|\nabla PU_2|^2dx-2\int_\Omega a(x)\nabla PU_1\nabla PU_2dx$$

Let us estimate the first term in (69). The estimate of the second term is similar. Let us choose η as in (49). We get

$$\begin{aligned}\int_\Omega a(x)|\nabla PU_1|^2dx&=-\int_\Omega \operatorname{div}\left(a(x)\nabla PU_1\right)PU_1dx\\&=-\int_\Omega\left(a(x)\Delta PU_1\right)PU_1dx-\int_\Omega\left\langle\nabla a,\nabla PU_1\right\rangle PU_1dx\\&=\int_\Omega a(x)U_1^{p-1}PU_1dx-\int_\Omega\left\langle\nabla a,\nabla PU_1\right\rangle PU_1dx\\&=\int_{B(\xi_1,\eta)}a(x)U_1^{p-1}PU_1dx+\int_{\Omega\setminus B(\xi_1,\eta)}a(x)U_1^{p-1}PU_1dx\\&\quad-\int_\Omega\left\langle\nabla a,\nabla PU_1\right\rangle PU_1dx\end{aligned}\tag{70}$$

By (33) we deduce for some $\beta,\sigma>0$

$$\int_\Omega\left\langle\nabla a,\nabla PU_1\right\rangle PU_1dx\le C\int_\Omega|\nabla PU_1|PU_1dx=O\left(\delta_1^{\frac{n-2}{n-1}+\beta}\right)=O\left(\epsilon^{1+\sigma}\right). \tag{71}$$

By Lemma A.2 we also deduce

$$\int_{\Omega\setminus B(\xi_1,\eta)}a(x)U_1^{p-1}PU_1dx=O\left(\left(\frac{\delta_1}{\epsilon}\right)^n\right) \tag{72}$$

and

(73)
$$\int\limits_{B(\xi_1,\eta)} a(x)U_1^{p-1}PU_1dx = \int\limits_{B(\xi_1,\eta)} a(x)U_1^{p}dx + \int\limits_{B(\xi_1,\eta)} a(x)U_1^{p-1}\left(PU_1 - U_1\right)dx.$$

The first term is estimated in Lemma C.1 and the second term is estimated in (64) of Lemma C.2.

It remains only to estimate the last term in (69).

$$\begin{aligned}\int\limits_{\Omega} a(x)\nabla PU_1\nabla PU_2dx &= -\int\limits_{\Omega} \operatorname{div}\left(a\nabla PU_1\right)PU_2dx \\ &= -\int\limits_{\Omega}\left(a\Delta PU_1\right)PU_2dx - \int\limits_{\Omega}\left\langle\nabla a,\nabla PU_1\right\rangle PU_2dx \qquad (74)\\ &= \int\limits_{\Omega} a(x)U_1^{p-1}PU_2dx - \int\limits_{\Omega}\left\langle\nabla a,\nabla PU_1\right\rangle PU_2dx.\end{aligned}$$

We have

$$\int\limits_{\Omega} a(x)U_1^{p-1}PU_2dx = \int\limits_{B(\xi_1,\eta)}\cdots + \int\limits_{\Omega\setminus B(\xi_1,\eta)}\cdots \tag{75}$$

and

(76)
$$\begin{aligned}&\int\limits_{\Omega\setminus B(\xi_1,\eta)} a(x)U_1^{p-1}PU_2dx \\ &= O\left(\delta_1^{\frac{n+2}{2}}\delta_2^{\frac{n-2}{2}}\int\limits_{\Omega\setminus B(\xi_1,\eta)}\frac{1}{|x-\xi_1|^{n+2}}\frac{1}{|x-\xi_2|^{n-2}}dx\right)\\ &= O\left(\frac{\delta_1^{\frac{n+2}{2}}\delta_2^{\frac{n-2}{2}}}{\eta^n}\int\limits_{\mathbb{R}^n\setminus B(0,1)}\frac{1}{|y|^{n+2}}\frac{1}{|y+\frac{\xi_1-\xi_2}{\eta}|^{n-2}}dy\right)\\ &= O\left(\frac{\delta_1^{\frac{n+2}{2}}\delta_2^{\frac{n-2}{2}}}{\eta^n}\right)\end{aligned}$$

The first term in (75) is estimated in (65) of Lemma C.2.

Finally, as in the proof of (71), from (33) we obtain

$$\int_{\Omega}\left\langle\nabla a,\nabla PU_2\right\rangle PU_1dx = O\left(\epsilon^{1+\sigma}\right), \tag{77}$$

since $0 < C_1 \leq \delta_2/\delta_1 \leq C_2$ on compact subsets of Λ.

We estimate the second term at the R.H.S. of (68). We write

$$\begin{aligned}\int_\Omega a(x)|V_{\mathbf{d},\mathbf{t}}|^p dx &= \int_\Omega a(x)|PU_1 - PU_2|^p dx \\ &= \int_\Omega a(x)\left(|PU_1 - PU_2|^p - |U_1|^p - |U_2|^p\right) dx \\ &\qquad + \int_\Omega a(x)\left(|U_1|^p + |U_2|^p\right) dx.\end{aligned} \tag{78}$$

The last two terms in (78) are estimated in Lemma C.1. Let us choose η as in (49).

We split the first integral as

$$\int_\Omega a(x)\left(|PU_1 - PU_2|^p - |U_1|^p - |U_2|^p\right) dx = \int_{B(\xi_1,\eta)} \cdots + \int_{B(\xi_2,\eta)} \cdots + \int_{\Omega\setminus(B(\xi_1,\eta)\cup B(\xi_2,\eta))} \cdots \tag{79}$$

From Lemma A.2 we deduce

$$\begin{aligned}&\int_{\Omega\setminus(B(\xi_1,\eta)\cup B(\xi_2,\eta))} a(x)\left(|PU_1 - PU_2|^p - |U_1|^p - |U_2|^p\right) dx \\ &\qquad = O\left(\int_{\Omega\setminus(B(\xi_1,\eta)\cup B(\xi_2,\eta))} \left(U_1^p + U_2^p\right) dx\right) = O\left(\frac{\delta_1^n}{\eta^n} + \frac{\delta_2^n}{\eta^n}\right).\end{aligned} \tag{80}$$

We now estimate the integral over $B(\xi_1,\eta)$.

$$\begin{aligned}&\int_{B(\xi_1,\eta)} a(x)\left(|PU_1 - PU_2|^p - |U_1|^p - |U_2|^p\right) dx \\ &= p\int_{B(\xi_1,\eta)} a(x)U_1^{p-1}\left(PU_1 - U_1 - PU_2\right) dx \\ &+ \frac{(p-1)p}{2}\int_{B(\xi_1,\eta)} a(x)|U_1 + \theta\left(PU_1 - U_1 - PU_2\right)|^{p-2}\left(PU_1 - U_1 - PU_2\right)^2 dx \\ &- \int_{B(\xi_1,\eta)} a(x)|U_2|^p dx \\ &= p\int_{B(\xi_1,\eta)} a(x)U_1^{p-1}\left(PU_1 - U_1 - PU_2\right) dx + I,\end{aligned} \tag{81}$$

where I is defined and estimated as

(82)

$$
\begin{aligned}
I &:= \frac{(p-1)p}{2} \int\limits_{B(\xi_1,\eta)} a(x)|U_1+\theta\left(PU_1-U_1-PU_2\right)|^{p-2}\left(PU_1-U_1-PU_2\right)^2 dx \\
&- \int\limits_{B(\xi_1,\eta)} a(x)|U_2|^p dx \\
&= O\left(\int\limits_{B(\xi_1,\eta)} U_1^{p-2}\left(PU_1-U_1\right)^2 dx\right) + O\left(\int\limits_{B(\xi_1,\eta)} U_1^{p-2}U_2^2 dx\right) \\
&+ O\left(\int\limits_{B(\xi_1,\eta)} |PU_1-U_1|^p\right) + O\left(\int\limits_{B(\xi_1,\eta)} |U_2|^p dx\right) \\
&= O\left(|U_1^{p-2}\left(PU_1-U_1\right)|_{B(\xi_1,\eta),\frac{2n}{n+2}}|PU_1-U_1|_{B(\xi_1,\eta),\frac{2n}{n-2}}\right) \\
&+ O\left(|U_1^{p-2}U_2|_{B(\xi_1,\eta),\frac{2n}{n+2}}|U_2|_{B(\xi_1,\eta),\frac{2n}{n-2}}\right) \\
&+ O\left(|PU_1-U_1|^p_{B(\xi_1,\eta),\frac{2n}{n-2}}\right) + O\left(|U_2|^p_{B(\xi_1,\eta),\frac{2n}{n-2}}\right) \\
&= O\left(\epsilon^{1+\sigma}\right),
\end{aligned}
$$

for some $\sigma > 0$, because of estimates (45), (46), (52) and (53).

The first term in (81) is estimated in (64) and (65) of Lemma C.2.

Finally, we estimate the integral over $B(\xi_2,\eta)$

(83)

$$
\begin{aligned}
&\int\limits_{B(\xi_2,\eta)} a(x)\left(|PU_1-PU_2|^p-|U_1|^p-|U_2|^p\right) dx \\
&= -p \int\limits_{B(\xi_2,\eta)} a(x)U_2^{p-1}\left(-PU_2+U_2+PU_1\right) dx \\
&+ \frac{(p-1)p}{2} \int\limits_{B(\xi_2,\eta)} a(x)|U_1+\theta\left(-PU_2+U_2+PU_1\right)|^{p-2}\left(-PU_2+U_2+PU_1\right)^2 dx \\
&- \int\limits_{B(\xi_2,\eta)} a(x)|U_1|^p dx \\
&= p \int\limits_{B(\xi_2,\eta)} a(x)U_2^{p-1}\left(PU_2-U_2-PU_1\right) dx + J,
\end{aligned}
$$

where J is estimated exactly as in (82), while the first term in (83) is estimated in (64) and (65) of Lemma C.2.

We collect all the previous estimates and we get the claim. $\square$

Lemma C.4. *The following estimate holds true:*

$$
\begin{aligned}
(84)\quad \frac{1}{p-\epsilon}\int\limits_{\Omega} a(x)|V_{\mathbf{s},\mathbf{d},\mathbf{t}}|^{p-\epsilon}dx &= \frac{1}{p}\int\limits_{\Omega} a(x)|V_{\mathbf{s},\mathbf{d},\mathbf{t}}|^{p}dx \\
&\quad + \epsilon\left[\frac{1}{p^2}\int\limits_{\Omega} a(x)|V_{\mathbf{s},\mathbf{d},\mathbf{t}}|^{p}dx - \frac{1}{p}\int\limits_{\Omega} a(x)|V_{\mathbf{s},\mathbf{d},\mathbf{t}}|^{p-1}\log|V_{\mathbf{s},\mathbf{d},\mathbf{t}}|dx\right] + o(\epsilon) \\
&= [a(s_1)+a(s_2)]\left(\frac{\gamma_1}{p^2} - \frac{\gamma_1\alpha_n}{p} - \frac{\gamma_3}{p}\right)\epsilon \\
&\quad + \frac{n-2}{2p}\gamma_1\left[a(s_1)\log\delta_1 + a(s_2)\log\delta_2\right]\epsilon + o(\epsilon).
\end{aligned}
$$

Proof. We argue exactly as in the proof of Lemma 3.2 of [9]. □

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