Self-focusing Multibump Standing Waves in Expanding Waveguides

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Abstract. Let M be a smooth k-dimensional closed submanifold of $\mathbb{R}^N$, $N \geq 2$, and let Ω_R be the open tubular neighborhood of radius 1 of the expanded manifold $M_R := \{Rx : x \in M\}$. For R sufficiently large we show the existence of positive multibump solutions to the problem

$$-\Delta u + \lambda u = f(u) \text{ in } \Omega_R, \qquad u = 0 \text{ on } \partial\Omega_R.$$

The function f is superlinear and subcritical, and $\lambda > -\lambda_1$, where λ_1 is the first Dirichlet eigenvalue of $-\Delta$ in the unit ball in $\mathbb{R}^{N-k}$.

1. Introduction

Let M be a compact k-dimensional smooth submanifold of $\mathbb{R}^N$ without boundary, $1 \leq k \leq N-1$, $N \geq 2$. For $R > 0$ define

$$\Omega_R := \bigcup_{x \in M} \left\{ Rx + v : v \in (T_x M)^\perp, \ |v| < 1 \right\},$$

where $T_x M$ is the tangent space of M at x, and $(T_x M)^\perp$ is its orthogonal complement in $\mathbb{R}^N$. Thus, for R large enough, Ω_R is the open tubular neighborhood of radius 1 of the expanded manifold $M_R := \{Rx : x \in M\}$.

We consider the problem

$$\begin{cases} -\Delta u + \lambda u = f(u) & \text{in } \Omega_R, \\ u = 0 & \text{on } \partial\Omega_R, \end{cases} \tag{1.1}$$

for $\lambda > -\lambda_1$, where λ_1 is the first Dirichlet eigenvalue of $-\Delta$ in the unit ball in $\mathbb{R}^{N-k}$.

Equation (1.1) appears in nonlinear optics and models standing waves in optical waveguides. The most interesting variant for applications that aspire to exploit the nonlinear properties of the material is the *self-focusing* case where $f(u)/u \to \infty$ as

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$|u| \to \infty$. A typical example is given by $f(u) = u^3$, modelling Kerr's effect. For more information on the physical background see for example [13].

Geometrically, if $k = 1$ then Ω_R is a *tubular guide*, i.e., an optical fiber. If $k \geq 2$ then we say that Ω_R is a *slab*, or, more specifically, a *shell* if $k = N - 1$.

For expanding annular shells, i.e., tubular neighborhoods of expanding spheres, there is a considerable body of existence results for multiple solutions that are symmetric with respect to some subgroup of $O(N)$, cf. [4–6,10–12,14]. The first existence result for multiple solutions we are aware of that does not depend on symmetries of the domain was given by Dancer and Yan [8] for expanding shells that bound convex domains, in the case of $\lambda = 0$.

Inspired by the article of Dancer and Yan we are interested in finding positive multibump solutions to (1.1) for large R, extending their result to more general domains and to the case of $\lambda \neq 0$.

Set $p_S := \infty$ if $N = 1, 2$ and $p_S := (N + 2)/(N - 2)$ if $N \geq 3$, and consider the following hypotheses on f:

(H1) $f \in C^1[0, \infty) \cap C^3(0, \infty)$;

(H2) there are $C > 0$ and $p_1, p_2 \in (1, p_S)$ such that $p_1 \leq p_2$ and

$$|f^{(k)}(u)| \leq C(|u|^{p_1 - k} + |u|^{p_2 - k})$$

for $k \in \{0, 1, 2, 3\}$ and $u > 0$;

(H3) $f(u) > 0$ for $u > 0$.

The strong differentiability conditions on f in (H1) and (H2) could be relaxed to C^1-differentiability, at the expense of extra conditions on Hölder continuity and bounds leading to the result of Lemma 2.2 below. Note that

$$f(0) = f'(0) = 0. \tag{1.2}$$

Clearly, $f(u) := |u|^{p-1}u$ satisfies (H1)–(H3) if $p \in (1, p_S)$.

Set

$$F(u) := \int_0^u f(s)\, ds \qquad \text{if } u \in \mathbb{R},$$

so

$$|F(u)| \leq C(|u|^{p_1+1} + |u|^{p_2+1}) \qquad \text{for all } u \in \mathbb{R} \tag{1.3}$$

by (H2). For a domain $\Omega \subseteq \mathbb{R}^N$ and the Dirichlet problem $-\Delta u + \lambda u = f(u)$ stated on Ω the variational (or energy) functional is given by

$$J_\Omega(u) := \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\, dx - \int_\Omega F(u)\, dx, \qquad u \in H_0^1(\Omega).$$

By (H1), (H2) and (1.3) J_Ω is well defined and twice continuously differentiable on $H_0^1(\Omega)$, with $\mathrm{D}^2 J_\Omega$ globally Hölder continuous on bounded subsets of $H_0^1(\Omega)$.

It is convenient to write a point in $\mathbb{R}^N$ as (ξ, η), where $\xi \in \mathbb{R}^k$ and $\eta \in \mathbb{R}^{N-k}$. Moreover, we denote by

$$\mathbb{L} := \{(\xi, \eta) \in \mathbb{R}^k \times \mathbb{R}^{N-k} : |\eta| < 1\}$$

the open cylinder or slab which is, locally, the limit of Ω_R as $R \to \infty$. Ground state solutions of the limit problem

$$\begin{cases} -\Delta u + \lambda u = f(u), \\ \quad u \in H_0^1(\mathbb{L}). \end{cases} \tag{1.4}$$

will serve as building blocks for multibump solutions. Assume that the following holds:

(H4) Problem (1.4) has a positive solution U which is radially symmetric in ξ and η separately, and which is nondegenerate, in the sense that the kernel of the space of solutions to the problem

$$-\Delta u + \lambda u = f'(U)u, \qquad u \in H_0^1(\mathbb{L}),$$

has dimension k.

Some examples of nonlinearities which satisfy (H4) are given in [7]. Note that every positive solution of (1.4) is radially symmetric in η and strictly decreasing in $|\eta|$, by [3, Theorem 1.2].

For each $x \in M_R$ we choose a linear isometry $A_x \in O(N)$ which maps the tangent space $T_x M$ onto $R^k \times \{0\}$ and $(T_x M)^\perp$ onto $\{0\} \times \mathbb{R}^{N-k}$, and we set

$$\mathbb{L}_x := \{x + A_x^{-1}(z) : z \in \mathbb{L}\}. \tag{1.5}$$

We consider U to be extended by 0 to all of $\mathbb{R}^N$, and for each $R > 0$ we define

$$U_{x,R}(y) := U(A_x(y - x)) \qquad \text{for } y \in \mathbb{R}^N. \tag{1.6}$$

Since U is radially symmetric in ξ and in η, the function $U_{x,R}$ does not depend on the choice of $A_x \in O(N)$ as long as it satisfies $A_x(T_x M) = \mathbb{R}^k \times \{0\}$.

Our result is the following.

Theorem 1.1. *Assume that* (H1)–(H4) *hold. Then, for each $n \in \mathbb{N}$ there exists $R_n > 0$ such that for every $R \geq R_n$ there are n points $x_{R,1}, \dots, x_{R,n} \in M_R$ and a positive solution u_R of* (1.1) *of the form*

$$u_R = \sum_{i=1}^{n} U_{x_{R,i},R} + o(1) \tag{1.7}$$

in $H^1(\mathbb{R}^N)$ as $R \to \infty$. Moreover, $|x_{R,i} - x_{R,j}| \to \infty$ as $R \to \infty$, if $i \neq j$.

In contrast to the results on expanding annular shells mentioned above, where in general the variational problem is considered in subspaces of functions that are symmetric with respect to a subgroup of $O(N)$, the proof of Theorem 1.1 rests on gluing rotated translates of the positive ground state solution whose existence is demanded by condition (H4). As $R \to \infty$ the possible number of bumps becomes arbitrarily large. This basic idea is the same as the one used by Dancer and Yan in [8]. Nevertheless, our proof is slightly different and simpler, and at the same time more detailed.

To prove our result we use a Lyapunov-Schmidt reduction argument and follow closely the approach of our previous paper [1]. As is typical for this method, the proof of existence of solutions with n bumps reduces to finding critical points of the natural variational functional $J_R := J_{\Omega_R}$ restricted to a finite dimensional manifold Σ_R with boundary, which can be understood as the subset of the configuration space of n points in M one obtains when subtracting a closed neighborhood of the collision set. An element of this manifold roughly marks the centers of mass of the n bumps to be glued together. The boundary of Σ_R consists of tuples that contain at least two centers of mass at a prescribed minimum distance. It turns out that the value of J_R decreases near the boundary of Σ_R, i.e. where two masses interact, so that J_R possesses an interior maximum on Σ_R, a global critical point.

In [1] instead we take $k = 1$ and build solutions out of bumps that are lined up along M with alternating signs. In that setting J_R *increases* near the boundary of Σ_R, so that a *minimum* exists in the interior. We also consider manifolds M with boundary and define Σ_R to consist of those elements from M^n that have centers of mass that stay clear of each other and of the boundary of M. Since J_R also increases as one bump approaches the boundary of M, there still exists a minimum of J_R in the interior of Σ_R.

When the domain is an annulus, i.e. it is a tubular neighborhood of the unit sphere, there is always a positive radial solution, which is obviously different from the multibump solutions that we find and which exists for all values of the exponent $p > 1$. It is natural to ask whether a similar solution exists also in the case of domains which are expanding annular shells, but different from the annulus. A positive answer in this direction is given by Bartsch, Clapp, Grossi and Pacella [2]. Moreover, an interesting open question is whether similar solutions exist in tubular neighborhoods of expanding k-dimensional manifolds ($0 < k < N$).

The outline of the paper is as follows: In Section 2 we compute the required estimates and in Section 3 we describe the finite dimensional reduction and prove Theorem 1.1.

2. Preliminary estimates

We recall some inequalities from [1].

Lemma 2.1. *Assume that $\mu_k > \bar{\mu} \geq 0$ for $k = 1, 2, 3$. Then there is $C > 0$ such that*

$$\int_{\mathbb{R}^N} e^{-\mu_1|x-x_1|} e^{-\mu_2|x-x_2|}\, dx \leq C e^{-\bar{\mu}|x_1-x_2|}$$

and

$$\int_{\mathbb{R}^N} e^{-\mu_1|x-x_1|} e^{-\mu_2|x-x_2|} e^{-\mu_3|x-x_3|}\, dx \leq C \exp\Big(-\bar{\mu} \min_{x\in\mathbb{R}^N} \sum_{k=1}^{3} |x - x_k|\Big)$$

for all $x_1, x_2, x_3 \in \mathbb{R}^N$.

Proof. See [1, Lemma 2.1]. □

Lemma 2.2. *There is $\alpha \in (1/2, 1]$ with the following property: Given $C \geq 1$ and $n \in \mathbb{N}$ there is a constant $\widetilde{C} = \widetilde{C}(\alpha, n, C) > 0$ such that for $u_1, u_2, \ldots, u_n \in \mathbb{R}$ with $|u_i| \leq C$ it holds that*

$$\left| f\left(\sum_{i=1}^{n} u_i\right) - \sum_{i=1}^{n} f(u_i) \right| \leq \widetilde{C} \sum_{i<j} |u_i u_j|^\alpha$$

and

$$\left| F\left(\sum_{i=1}^{n} u_i\right) - \sum_{i=1}^{n} F(u_i) - \sum_{i \neq j} f(u_i) u_j \right| \leq \widetilde{C} \left(\sum_{i<j} |u_i u_j|^{2\alpha} + \sum_{i<j<k} |u_i u_j u_k|^{2/3} \right).$$

Proof. See [1, Lemma 2.2]. □

Set $\mathbb{B}_s^m := \{x \in \mathbb{R}^m : |x| < s\}$. Let $\lambda_{1,s}$ be the first Dirichlet eigenvalue of $-\Delta$ in $\mathbb{B}_s^{N-k}$ and $\vartheta_{1,s}$ be the corresponding L^2-normalized positive eigenfunction. For the solution of the limit problem (1.4) the following decay estimates hold.

Lemma 2.3. *There are constants $c_1, c_2 > 0$ such that*

$$c_1 |\xi|^{-\frac{k-1}{2}} e^{-\mu|\xi|} \vartheta_{1,1}(\eta) \leq |U(\xi, \eta)| \leq c_2 |\xi|^{-\frac{k-1}{2}} e^{-\mu|\xi|} \vartheta_{1,1}(\eta) \quad \textit{for all } (\xi, \eta) \in \mathbb{L},$$

where $\mu := \sqrt{\lambda + \lambda_1}$.

Proof. This follows in the same way as Eq. (7) in [7, Theorem 4]. □

Let $\mathbb{L}_s := \{(\xi, \eta) \in \mathbb{R}^k \times \mathbb{R}^{N-k} : |\eta| < s\}$. The bottom of the spectrum of $-\Delta$ in $L^2(\mathbb{L}_s)$ with Dirichlet boundary conditions coincides with $\lambda_{1,s}$, cf. [1, Lemma 2.5], whose proof extends to the present case.

We fix $r > 1$ such that $\lambda_{1,r} + \lambda > 0$. For $R > 0$, $x \in M_R$ and $s \in [1, r]$, let

$$\mathbb{L}_{s,x} := \{x + A_x^{-1}(z) : z \in \mathbb{L}_s\}$$

with A_x as in (1.5). Note that the first eigenvalue of $-\Delta$ in $H_0^1(\Omega_R \cap \mathbb{L}_{s,x})$ satisfies $\lambda_1(\Omega_R \cap \mathbb{L}_{s,x}) + \lambda > 0$ for large R, since $\Omega_R \cap \mathbb{L}_{s,x}$ is an open bounded subset of $\mathbb{L}_{r,x}$. We write $V_{x,s,R}$ for the unique solution to the problem

$$\begin{cases} -\Delta u + \lambda u = f(U_{x,R}) & \text{in } \Omega_R \cap \mathbb{L}_{s,x}, \\ u = 0 & \text{on } \partial(\Omega_R \cap \mathbb{L}_{s,x}), \end{cases} \tag{2.1}$$

with $U_{x,R}$ as in (1.6). Assumption (H3) and the maximum principle, which applies by the argument in [1, Remark 2.8], yield that $V_{x,s,R}$ is positive for R large enough. We consider $V_{x,s,R}$ as extended by 0 to $\mathbb{R}^N$. When $s = 1$ we omit it from the notation and write λ_1, $\mathbb{L}_x$, $V_{x,R}$ instead of $\lambda_{1,1}$, $\mathbb{L}_{1,x}$, $V_{x,1,R}$.

The following decay estimates hold.

Lemma 2.4. *For each $s \in [1, r)$ there are positive constants c_3, c_4 and R_0, independent of $x \in M_R$, such that all quantities*

$$|U_{x,R}(y)|, \quad |\nabla U_{x,R}(y)|, \quad |V_{x,s,R}(y)|, \quad |\nabla V_{x,s,R}(y)|,$$

are bounded by $c_3 e^{-c_4|y-x|}$ for all $R \geq R_0$ and almost all $y \in \mathbb{R}^N$. Moreover,

$$\left|D^2 U_{x,R}(y)\right| \quad \textit{and} \quad \left|D^2 V_{x,s,R}(y)\right|$$

are bounded uniformly in $\mathbb{L}_x$ *and* $\Omega_R \cap \mathbb{L}_{s,x}$ *respectively, independently of* $R \geq R_0$.

Proof. Lemma 2.3, together with standard regularity estimates, yields the estimates for $U_{x,R}$ and its derivatives.
To prove the estimates for $V_{x,s,R}$ we assume without loss of generality that $x = 0$ and that $\mathbb{R}^k \times \{0\}$ is the tangent space to M_R at 0. Then there exists $\tilde{c}_s > 0$ such that $\vartheta_{1,r}(\eta) \geq \tilde{c}_s$ for all $\eta \in \mathbb{B}_s^{N-k}$. We write $y \in \mathbb{L}_s$ as (ξ, η) with $\xi \in \mathbb{R}^k$ and $\eta \in \mathbb{B}_s^{N-k}$, and set

$$W(y) := e^{-\nu|\xi|}\vartheta_{1,r}(\eta)$$

where ν is a small positive constant, independent of R, which will be fixed next. A straightforward computation gives

$$\begin{aligned} -\Delta W(y) + \lambda W(y) &= \left(\frac{(N-1)\,\nu}{|\xi|} - \nu^2 + \lambda_{1,r} + \lambda \right) W(y) \\ &> \left(\lambda_{1,r} + \lambda - \nu^2 \right) \tilde{c}_s e^{-\nu|\xi|}. \end{aligned}$$

Since $\lambda_{1,r} + \lambda > 0$ we have that $\lambda_{1,r} + \lambda - \nu^2 > 0$ if ν is small enough. On the other hand, assumption (H2) on f together with Lemma 2.3 yield that

$$f(U_{x,R}) \leq \tilde{c}_1 e^{-\mu p_1|\xi|},$$

for some large enough $\tilde{c}_1 > 0$. By comparison with equation (2.1) we obtain that $V_{x,s,R} \leq \tilde{c}_2 W$ with $\tilde{c}_2 := \tilde{c}_1 \tilde{c}_s^{-1} \left(\lambda_{1,r} + \lambda - \nu^2 \right)^{-1}$ and, hence, the exponential bound on $V_{x,s,R}$. Regularity estimates using the results in [9] yield the estimates for its derivatives. Note that the boundary of $\Omega_R \cap \mathbb{L}_{s,x}$ is Lipschitz and satisfies an exterior ball condition, uniformly as $R \to \infty$. □

Lemma 2.5. *If* $s \in [1, r)$ *and* $p \in (0, \infty)$ *then we have the following asymptotic estimates as* $R \to \infty$*, independently of* $x \in M_R$*:*

$$\int_{\mathbb{R}^N} |V_{x,s,R} - U_{x,R}|^p dy = O(R^{-\min\{p,1\}}), \tag{2.2}$$

$$\int_{\mathbb{R}^N} |\nabla V_{x,s,R} - \nabla U_{x,R}|^2 dy = O(R^{-1}), \tag{2.3}$$

$$\int_{\mathbb{R}^N} |F(V_{x,s,R}) - F(U_{x,R})| dy = O(R^{-1}), \tag{2.4}$$

$$\int_{\mathbb{R}^N} |f(V_{x,s,R}) - f(U_{x,R})|^p dy = O(R^{-\min\{p,1\}}). \tag{2.5}$$

Proof. Let x be a point on the manifold M. After translation and rotation we may assume that $x = 0$ and that $\mathbb{R}^k \times \{0\}$ is the tangent space to M at 0. Since M is compact we may find $\delta, \rho > 0$, independent of x, and a smooth map $h : \mathbb{B}_\rho^k \to \mathbb{B}_\delta^{N-k}$ such that

$$M \cap \left(\mathbb{B}_\rho^k \times \mathbb{B}_\delta^{N-k} \right) = \{(\xi, h(\xi)) : \xi \in \mathbb{B}_\rho^k\}$$

whose derivatives up to the order 3 are bounded independently of $\xi \in \mathbb{B}_\rho^k$ and $x \in M$. Setting $h_R(\xi) := Rh(\xi/R)$ we have that

$$\widetilde{M}_R := M_R \cap \left(\mathbb{B}_{\rho R}^k \times \mathbb{B}_{\delta R}^{N-k} \right) = \{(\xi, h_R(\xi)) : \xi \in \mathbb{B}_{\rho R}^k\}.$$

Following the argument given in [1, Lemma 3.2] one shows that there is a constant C, independent of x, such that

$$|h_R(\xi)| \leq \frac{C\,|\xi|^2}{R}$$

for all $\xi \in \mathbb{B}^k_{\rho R}$, and

$$\{\xi\} \times \mathbb{B}^{N-k}_1(h_R(\xi)) \subset \left(\{\xi\} \times \mathbb{R}^{N-k}\right) \cap \widetilde{\Omega}_R \subset \{\xi\} \times \mathbb{B}^{N-k}_{1+C(1+|\xi|^2)/R^2}(h_R(\xi))$$

for all $\xi \in \mathbb{B}^k_{\rho R-1}$ and R large enough, where $\widetilde{\Omega}_R$ is the tubular neighborhood of $\widetilde{M}_R$ of radius 1. Consider the set

$$Q_R := \mathbb{B}^k_{R^{1/4}} \times \mathbb{B}^{N-k}_s \subset \mathbb{L}_s$$

and decompose $\mathbb{R}^N$ as the union of the sets

$$\mathbb{R}^N \setminus Q_R, \qquad Q_R \cap (\Omega_R \setminus \mathbb{L}), \qquad Q_R \cap (\mathbb{L} \setminus \Omega_R), \qquad Q_R \cap \mathbb{L} \cap \Omega_R.$$

Note that the integrals over $Q_R \setminus (\mathbb{L} \cup \Omega_R)$ are zero. Using Lemma 2.4 and following closely the proof of [1, Lemma 3.2], with the obvious modifications, one obtains (2.2), (2.3), (2.4), (2.5). □

Recall that

$$J_R(u) = \frac{1}{2}\int_{\Omega_R} (|\nabla u|^2 + \lambda u^2)dx - \int_{\Omega_R} F(u)dx, \qquad u \in H^1_0(\Omega_R),$$
$$J_{\mathbb{L}}(u) = \frac{1}{2}\int_{\mathbb{L}} (|\nabla u|^2 + \lambda u^2)dx - \int_{\mathbb{L}} F(u)dx, \qquad u \in H^1_0(\mathbb{L}),$$

are the energy functionals associated to problems (1.1) and (1.4) respectively.

Lemma 2.6.

$$\sup_{x \in M_R} \|V_{x,R} - U_{x,R}\|_{H^1(\mathbb{R}^N)} = O(R^{-1/2}), \tag{2.6}$$
$$\sup_{x \in M_R} |J_R(V_{x,R}) - J_{\mathbb{L}}(U)| = O(R^{-1}), \tag{2.7}$$
$$\sup_{x \in M_R} \|\nabla J_R(V_{x,R})\|_{H^1_0(\Omega_R)} = O(R^{-1/2}), \tag{2.8}$$

as $R \to \infty$.

Proof. The first two asymptotic estimates follow immediately from Lemma 2.5. To prove the third one we choose $s \in (1, r)$ and a cut-off function $\chi \in C^\infty(\mathbb{R}^{N-k})$ with $\chi(\eta) = 1$ if $|\eta| \leq 1$ and $\chi(\eta) = 0$ if $|\eta| \geq s$. Fix R and $x \in M_R$. Assuming that $x = 0$ and that $\mathbb{R}^k \times \{0\}$ is the tangent space to M_R at 0, we write $v \in H^1_0(\Omega_R)$ as $v = v_1 + v_2$ where $v_1(\xi, \eta) := \chi(\eta)v(\xi, \eta)$. Then $v_1 \in H^1_0(\Omega_R \cap \mathbb{L}_{s,x})$, $\text{supp}(v_2) \subset \Omega_R \setminus \mathbb{L}_x$ and there exists a constant c_s, independent of R and x, such that $\|v_1\|_{H^1(\mathbb{R}^N)} \leq c_s \|v\|_{H^1(\mathbb{R}^N)}$

for all $v \in H_0^1(\Omega_R)$. Applying the definition of $V_{x,s,R}$ and Lemma 2.5 we obtain

$$\begin{aligned}|DJ_R(V_{x,R})v| &= |DJ_R(V_{x,R})v_1| \\ &\le |DJ_R(V_{x,s,R})v_1| + |DJ_R(V_{x,R})v_1 - DJ_R(V_{x,s,R})v_1| \\ &\le \left|\int_{\mathbb{R}^N} (f(U_{x,R}) - f(V_{x,s,R}))\, v_1\right| + O(R^{-1/2})\, \|v_1\|_{H^1(\mathbb{R}^N)} \\ &\le O(R^{-1/2})\, \|v\|_{H^1(\mathbb{R}^N)}\,,\end{aligned}$$

as claimed. □

For $m = 1, 2$ we consider functions $g_m : \mathbb{R}^+ \to \mathbb{R}^+$ (to be fixed later) satisfying

$$g_2 < g_1, \tag{2.9}$$

$$g_m(R) \to \infty \qquad \text{as } R \to \infty, \tag{2.10}$$

$$g_m(R) = o(R) \qquad \text{as } R \to \infty. \tag{2.11}$$

Let $D_{m,R}$ be the set of all points $(x_1, \dots, x_n) \in (M_R)^n$ such that there exist $i, j \in \{1, \dots, n\}$ with $i \neq j$ and $|x_i - x_j| \le g_m(R)$, and let

$$\mathcal{U}_{m,R} := (M_R)^n \smallsetminus D_{m,R}.$$

Then $\mathcal{U}_{1,R}$ and $\mathcal{U}_{2,R}$ are open subsets of $(M_R)^n$ with $\overline{\mathcal{U}_{1,R}} \subset \mathcal{U}_{2,R}$. For $X = (x_1, \dots, x_n) \in \mathcal{U}_{2,R}$ we define $\varphi_R : \mathcal{U}_{2,R} \to H_0^1(\Omega_R)$ by

$$\varphi_R(X) := \sum_{i=1}^n V_{x_i,R}. \tag{2.12}$$

Proposition 2.7. *Let α be as in Lemma 2.2 and fix $\gamma \in (1/2, \alpha)$. Then*

$$\sup_{X \in \mathcal{U}_{2,R}} \|\nabla J_R(\varphi_R(X))\|_{H_0^1(\Omega_R)} = O(e^{-\gamma\mu g_2(R)}) + O(R^{-1/2})$$

as $R \to \infty$.

Proof. Fix $X = (x_1, \dots, x_n) \in \mathcal{U}_{2,R}$. To simplify notation we write $U_i := U_{x_i,R}$ and $V_i := V_{x_i,R}$. If $v \in H_0^1(\Omega_R)$ satisfies $\|v\|_{H_0^1(\Omega_R)} = 1$, using Lemmas 2.6, 2.2, 2.5, 2.3 and 2.1, in this order, we obtain

$$\begin{aligned}\Big|DJ_R(\varphi_R(X))v\Big| &= \left|\sum_{i=1}^n DJ_R(\varphi_R(V_i))v + \int_{\Omega_R}\left(\sum_{i=1}^n f(V_i) - f\left(\sum_{i=1}^n V_i\right)\right)v\right| \\ &\le \sum_{i=1}^n \|\nabla J_R(V_{x,R})\|_{H_0^1(\Omega_R)} + \left(\int_{\Omega_R}\left|\sum_{i=1}^n f(V_i) - f\left(\sum_{i=1}^n V_i\right)\right|^2\right)^{1/2} \\ &\le O(R^{-1/2}) + C\sum_{i<j}\left(\int_{\Omega_R}|V_iV_j|^{2\alpha}\right)^{1/2} \\ &= O(R^{-1/2}) + C\sum_{i<j}\left(\int_{\Omega_R}|U_iU_j|^{2\alpha}\right)^{1/2} \\ &= O(R^{-1/2}) + O(e^{-\gamma\mu g_2(R)}).\end{aligned}$$

These estimates are independent of the choice of X. □

Set

$$E := J_{\mathbb{L}}(U)$$

and define

$$g_3(R) := g_1(R)^{-\frac{k-1}{2}} e^{-\mu g_1(R)} \qquad \text{for } R > 0.$$

Proposition 2.8. *There exists $\beta > 0$ such that*

$$\sup_{X \in \partial \mathcal{U}_{1,R}} J_R(\varphi_R(X)) \leq nE - \beta g_3(R) + o(g_3(R)) + O(R^{-1/2})$$

as $R \to \infty$.

Proof. If $X \in \partial \mathcal{U}_{1,R}$ then $|x_i - x_j| \geq g_1(R)$ for all $i, j \in \{1, \ldots, n\}$ and there exist $i_0 \neq j_0$ with $|x_{i_0} - x_{j_0}| = g_1(R)$. We write $U_i := U_{x_i,R}$ and $V_i := V_{x_i,R}$. By Lemma 2.3 we may choose $\varepsilon, \rho > 0$ such that $f(U_{i_0}) > \varepsilon$ and $U_{j_0} \geq C g_3(R)$ in $B_\rho(x_{i_0}) := \{y \in R^N : |y - x_{i_0}| < \rho\}$ for R large enough, independently of $X \in \partial \mathcal{U}_{1,R}$. Hence, for some $\beta > 0$ and large enough R we have that

$$\int_{\mathbb{R}^N} f(U_{i_0}) U_{j_0} \geq \beta g_3(R). \tag{2.13}$$

Since U_i and V_i are uniformly bounded, using Lemma 2.2, estimate (2.2) and Lemmas 2.3 and 2.1 we obtain, as in the proof of [1, Prop. 3.5],

$$\begin{aligned} &\left| \int_{\Omega_R} \left[F\left(\sum_i V_i \right) - \sum_i F(V_i) \right] - \sum_{i \neq j} \int_{\Omega_R} f(V_i) V_j \right| \\ &\qquad \leq C \sum_{i<j} \int_{\Omega_R} |V_i V_j|^{2\alpha} + C \sum_{i<j<k} \int_{\Omega_R} |V_i V_j V_k|^{2/3} \\ &\qquad = C \sum_{i<j} \int_{\Omega_R} |U_i U_j|^{2\alpha} + C \sum_{i<j<k} \int_{\Omega_R} |U_i U_j U_k|^{2/3} + O(R^{-2/3}) \\ &\qquad = o(g_3(R)) + O(R^{-2/3}). \end{aligned} \tag{2.14}$$

Therefore, using estimates (2.7), (2.14), (2.2), (2.5) and (2.13) we conclude that

$$\begin{aligned}
J_R(\varphi_R(X)) &= \sum_i J_R(V_i) + \frac{1}{2}\sum_{i\neq j}\int_{\Omega_R}(\nabla V_i\cdot\nabla V_j + \lambda V_i V_j)\\
&+ \int_{\Omega_R}\Big[\sum_i F(V_i) - F\Big(\sum_i V_i\Big)\Big]\\
&= nE + \frac{1}{2}\sum_{i\neq j}\int_{\Omega_R} f(U_i)V_j - \sum_{i\neq j}\int_{\Omega_R} f(V_i)V_j\\
&+ o(g_3(R)) + O(R^{-1/2})\\
&= nE - \frac{1}{2}\sum_{i\neq j}\int_{\Omega_R} f(U_i)U_j + o(g_3(R)) + O(R^{-1/2})\\
&\le nE - \frac{1}{2}\int_{\Omega_R} f(U_{i_0})U_{j_0} + o(g_3(R)) + O(R^{-1/2})\\
&\le nE - \beta g_3(R) + o(g_3(R)) + O(R^{-1/2}),
\end{aligned}$$

as claimed. □

Proposition 2.9. *As* $R\to\infty$,

$$\sup_{X\in\mathcal{U}_{1,R}} J_R(\varphi_R(X)) \ge nE + o(g_3(R)) + O(R^{-1/2}).$$

Proof. We fix n distinct points $x_1,\dots,x_n\in M$ and set $X_R := (Rx_1,\dots,Rx_n)$. Then $X_R\in\mathcal{U}_{1,R}$ for R large enough because of (2.11). As in the proof of Proposition 2.8 we have that

$$J_R(\varphi_R(X)) = nE - \frac{1}{2}\sum_{i\neq j}\int_{\Omega_R} f(U_i)U_j + o(g_3(R)) + O(R^{-1/2}).$$

Let $\delta\in(0,\min_{i\neq j}|x_i-x_j|)$ and $\bar\mu\in(0,\mu)$. Lemmas 2.3 and 2.1 yield

$$\int_{\Omega_R} f(U_i)U_j = O(e^{-\bar\mu\delta R}) = o(e^{-\mu g_1(R)})$$

for all $i\neq j$, and our claim follows. □

3. Finite dimensional reduction and proof of Theorem 1.1

Let $\Sigma_R := \varphi_R(\mathcal{U}_{2,R})$. The map φ_R is a C^2-immersion of $\mathcal{U}_{2,R}$ into $H_0^1(\Omega_R)$ (cf. [1, Lemma 4.1]) but it is not injective if $n>1$. Two points in $\mathcal{U}_{2,R}$ have the same image under φ_R if and only if one of them is obtained from the other by a permutation of coordinates in M_R^n. Since the group of permutations acts freely on $\mathcal{U}_{2,R}$, the set Σ_R is a C^2-submanifold of $H_0^1(\Omega_R)$.

For $u\in\Sigma_R$ we denote by $P_{u,R}$ the orthogonal projection onto the normal space $N_u\Sigma_R := (T_u\Sigma_R)^\perp$ to Σ_R at u. For each $u\in H_0^1(\Omega_R)$ we consider $D^2J_R(u)$ as an element of $\mathcal{L}(H_0^1(\Omega_R))$, i.e. as the derivative of the map $\nabla J_R : H_0^1(\Omega_R)\to H_0^1(\Omega_R)$ at the point u.

The next two lemmas are proved in the same way as the analogous statements in [1].

Lemma 3.1. *There are $r_0 > 0$ and $R_1 \geq 1$ such that for $R \geq R_1$ and for every $u \in \Sigma_R$ there is a unique $v_u \in u + N_u\Sigma_R$ which satisfies $\|u - v_u\|_{H_0^1(\Omega_R)} < r_0$ and $P_{u,R}\nabla J_R(v_u) = 0$. Moreover,*

$$\|u - v_u\|_{H_0^1(\Omega_R)} = O(\|\nabla J_R(u)\|_{H_0^1(\Omega_R)}) \tag{3.1}$$

and

$$|J_R(u) - J_R(v_u)| = O(\|\nabla J_R(u)\|^2_{H_0^1(\Omega_R)}) \tag{3.2}$$

as $R \to \infty$ independently of $u \in \Sigma_R$, and the operator $P_{u,R}D^2 J_R(v_u)|_{N_u\Sigma_R}$ is invertible in $\mathcal{L}(N_u\Sigma_R)$.

Proof. See [1, Lemma 4.3]. □

We now fix $r_0 > 0$ and $R_1 \geq 1$ as in Lemma 3.1. For $R \geq R_1$ let $\psi_R : \mathcal{U}_{2,R} \to H_0^1(\Omega_R)$ be given by $\psi_R(X) := v_u$, where $u := \varphi_R(X)$ and v_u is given by Lemma 3.1. Define $G_R : \mathcal{U}_{2,R} \to \mathbb{R}$ by

$$G_R := J_R \circ \psi_R.$$

Lemma 3.2. *For $R \geq R_1$ the map G_R is of class C^1. If $X \in \mathcal{U}_{2,R}$ is a critical point of G_R then $\psi_R(X)$ is a critical point of J_R.*

Proof. See [1, Lemma 4.4]. □

Proof of Theorem 1.1. By Lemma 3.2 it suffices to show that G_R has a local maximum in $\mathcal{U}_{2,R}$. Propositions 2.7, 2.8 and 2.9, together with estimate (3.2), yield the following inequalities:

$$\max G_R(\partial\mathcal{U}_{1,R}) \leq nE - \beta g_3(R) + o(g_3(R)) + O(R^{-1/2}) + O(e^{-2\gamma\mu g_2(R)}),$$
$$\max G_R(\overline{\mathcal{U}_{1,R}}) \geq nE + o(g_3(R)) + O(R^{-1/2}) + O(e^{-2\gamma\mu g_2(R)}),$$

with $\beta > 0$ and $\gamma > 1/2$. We choose

$$g_1(R) := \frac{1}{4\mu}\log R \qquad \text{and} \qquad g_2(R) := \left(\frac{1}{2} + \frac{1}{4\gamma}\right) g_1(R).$$

They clearly satisfy (2.9), (2.10) and (2.11). Note that

$$R^{-1/2} = o(g_3(R)) \qquad \text{and} \qquad e^{-2\gamma\mu g_2(R)} = o(g_3(R)).$$

Therefore,

$$\max G_R(\overline{\mathcal{U}_{1,R}}) > \max G_R(\partial\mathcal{U}_{1,R})$$

for large R. Hence G_R has a local maximum X_R in $\mathcal{U}_{2,R}$. Estimate (3.1), together with Lemma 2.6, proves (1.7). Finally, $X_R = (x_{R,1}, \ldots, x_{R,n}) \in \mathcal{U}_{1,R}$ and (2.10) yield $|x_{R,i} - x_{R,j}| \to \infty$ as $R \to \infty$, if $i \neq j$. □

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