ALTERNATING SIGN MULTIBUMP SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS IN EXPANDING TUBULAR DOMAINS

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Abstract. Let Γ denote a smooth simple curve in $\mathbb{R}^N$, $N \geq 2$, possibly with boundary. Let Ω_R be the open normal tubular neighborhood of radius 1 of the expanded curve $R\Gamma := \{Rx \mid x \in \Gamma \smallsetminus \partial\Gamma\}$. Consider the superlinear problem $-\Delta u + \lambda u = f(u)$ on the domains Ω_R, as $R \to \infty$, with homogeneous Dirichlet boundary condition. We prove the existence of multibump solutions with bumps lined up along $R\Gamma$ with alternating signs. The function f is superlinear at 0 and at ∞, but it is not assumed to be odd.

If the boundary of the curve is nonempty our results give examples of contractible domains in which the problem has multiple sign changing solutions.

1. Introduction

Let $\gamma \in C^3([0,1], \mathbb{R}^N)$, $N \geq 2$, be a curve without self-intersections except possibly for $\gamma(0) = \gamma(1)$. In this case we also assume that $\dot\gamma(0) = \dot\gamma(1)$. For $R > 0$ define

$$\Omega_R := \operatorname{int} \bigcup_{t\in[0,1]} \{R\gamma(t) + v \mid v \in \mathbb{R}^N,\ |v| < 1,\ \dot\gamma(t)\cdot v = 0\}, \tag{1.1}$$

where $\operatorname{int}(X)$ denotes the interior of X in $\mathbb{R}^N$. Thus, for R large enough, Ω_R is the tubular neighborhood of radius 1 of the 1-dimensional submanifold Γ_R of $\mathbb{R}^N$ defined as

$$\Gamma_R := \begin{cases} \{R\gamma(t) \mid t \in [0,1]\}, & \text{if } \gamma(0) = \gamma(1), \\ \{R\gamma(t) \mid t \in (0,1)\}, & \text{if } \gamma(0) \neq \gamma(1). \end{cases}$$

We are interested in finding solutions to the problem

$$\begin{cases} -\Delta u + \lambda u = f(u) & \text{in } \Omega_R, \\ u = 0 & \text{on } \partial\Omega_R, \end{cases} \tag{1.2}$$

for R large enough.

Let $\lambda_{1,1}$ be the first eigenvalue of the Laplace operator $-\Delta$ in the unit ball in $\mathbb{R}^{N-1}$ with Dirichlet boundary conditions. Set $p_S := \infty$ if $N = 1, 2$ and $p_S := (N+2)/(N-2)$ if $N \geq 3$. We make the following assumptions:

(H1) $\lambda > -\lambda_{1,1}$.
(H2) $f \in C^1(\mathbb{R}) \cap C^3(\mathbb{R}\backslash\{0\})$.

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(H3) There are $C > 0$ and $p_1, p_2 \in (1, p_S)$ such that $p_1 \le p_2$ and

$$|f^{(k)}(u)| \le C(|u|^{p_1-k} + |u|^{p_2-k})$$

for $k \in \{0,1,2,3\}$ and $u \neq 0$.

(H4) $f(u)u > 0$ for all $u \neq 0$.

Note that

$$f(0) = f'(0) = 0. \tag{1.3}$$

For example, the standard nonlinearity $f(u) := |u|^{p-1}u$ satisfies (H1)-(H4) if $p \in (1, p_S)$.

We write a point in $\mathbb{R}^N$ as (ξ, η), with $\xi \in \mathbb{R}$ and $\eta \in \mathbb{R}^{N-1}$, and denote the cylinder in $\mathbb{R}^N$ of radius 1 around the ξ-axis by

$$\mathbb{L} := \{(\xi, \eta) \in \mathbb{R}^N \mid |\eta| < 1\}.$$

Locally, $\mathbb{L}$ is the limit domain of Ω_R as $R \to \infty$. So we consider the limit problem

$$\begin{cases} -\Delta u + \lambda u = f(u), \\ u \in H_0^1(\mathbb{L}). \end{cases} \tag{1.4}$$

By Lemma 2.5 below, the operator $-\Delta + \lambda$ with Dirichlet boundary conditions in $L^2(\mathbb{L})$ has a positive spectrum. If f satisfies an Ambrosetti-Rabinowitz type condition the mountain pass theorem, together with the translation invariance in the ξ-direction and concentration compactness, yields a positive and a negative solution to (1.4), having minimal energy in their respective cones. We add the following assumption:

(H5) Problem (1.4) has a positive solution U^+ and a negative solution U^- which are nondegenerate, in the sense that the solution space of the linearized problem

$$-\Delta u + \lambda u = f'(U^\pm)u, \qquad u \in H_0^1(\mathbb{L}),$$

has dimension one.

Note that the solution space of the linearized problem must have at least dimension one, due to the invariance under translations. Hypothesis (H5) requires that these are the only elements in the kernel of the linearization. This condition is not easy to check, even for the standard nonlinearity $f(u) := u^p$. For this f, Dancer showed in [9] that (H5) holds true either for $\lambda = 0$ and almost every $p \in (1, p_S)$, or for almost every $\lambda \in (0, \infty)$ and every $p \in (1, p_S)$.

By [4, Theorem 1.2] the solutions $U^\pm$ are radially symmetric in η and decreasing in $|\eta|$. Moreover, by [5, Theorem 6.2], after a translation in the ξ-direction, we may assume that they are also even in ξ and decreasing in $|\xi|$. It follows that they have a unique extremal point at 0. We extend $U^\pm$ to all of $\mathbb{R}^N$ by setting them as 0 outside of $\mathbb{L}$.

For each $x \in \Gamma_R$ we choose a linear isometry A_x which maps the tangent space of Γ_R at x onto $\mathbb{R} \times \{0\}$ and its orthogonal complement onto $\{0\} \times \mathbb{R}^{N-1}$, and we define

$$U^\pm_{x,R}(y) := U^\pm(A_x(y - x)) \qquad \text{for all } y \in \mathbb{R}^N. \tag{1.5}$$

Since $U^\pm$ is radially symmetric in ξ and in η, the function $U^\pm_{x,R}$ is independent of the choice of A_x.

The parametrization γ induces an orientation on Γ_R which allows to give an order to every finite set of points in Γ_R. We shall say that $(x_1, \ldots, x_n) \in (\Gamma_R)^n$ is an *n-chain* in Γ_R if there exist $0 \leq t_1 < t_2 < \cdots < t_n < 1$ such that

$$x_i = R\gamma(t_i) \qquad \text{for } i = 1, 2, \ldots, n. \tag{1.6}$$

If $\gamma(0) = \gamma(1)$ a circular shift $(x_i, \ldots, x_n, x_1, \ldots, x_{i-1})$ of an n-chain will also be called an n-chain. We shall prove the following results.

Theorem 1.1. *Assume that $\gamma(0) = \gamma(1)$. Suppose also that* (H1)-(H5) *hold. For each $k \in \mathbb{N}$ there exists $R_k > 0$ such that for every $R \geq R_k$ there are a $2k$-chain $(x_{R,1}, x_{R,2}, \ldots, x_{R,2k}) \in (\Gamma_R)^{2k}$ and a solution u_R of* (1.2) *such that*

$$u_R = \sum_{i=1}^{k} (U^+_{x_{R,2i-1},R} + U^-_{x_{R,2i},R}) + o(1) \tag{1.7}$$

in $H^1(\mathbb{R}^N)$ as $R \to \infty$. Moreover, $|x_{R,i} - x_{R,j}| \to \infty$ as $R \to \infty$, if $i \neq j$.

Theorem 1.2. *Assume that $\gamma(0) \neq \gamma(1)$. Suppose also that* (H1)-(H5) *hold. For each $n \in \mathbb{N}$, $n \geq 2$, there exists $R_n > 0$ such that for every $R \geq R_n$ there are an n-chain $(x_{R,1}, x_{R,2}, \ldots, x_{R,n}) \in (\Gamma_R)^n$ and a solution u_R of* (1.2) *such that*

$$u_R = \sum_{i=1}^{k} (U^+_{x_{R,2i-1},R} + U^-_{x_{R,2i},R}) + (n-2k)U^+_{x_{R,n},R} + o(1) \tag{1.8}$$

in $H^1(\mathbb{R}^N)$ as $R \to \infty$, where k is the largest integer smaller than or equal to $n/2$. Moreover, as $R \to \infty$, $|x_{R,i} - x_{R,j}| \to \infty$ if $i \neq j$, and $\text{dist}(x_{R,i}, \partial\Gamma_R) \to \infty$ *for all i.*

All solutions constructed in Theorems 1.1 and 1.2 change sign. If γ is a closed curve these solutions have an even number of bumps with alternating signs along the curve, whereas in the open-end case $\gamma(0) \neq \gamma(1)$ the number of alternating bumps may be even or odd. Note that the term $(n - 2k)$ in Theorem 1.2 is 0 if n is even, and it is 1 if n is odd. In the first case we have a positive bump at one end and a negative bump at the other end of the domain, and in the second case we have positive bumps at both ends. Of course, applying Theorem 1.2 with $f(u)$ replaced by $-f(-u)$ and then multiplying the obtained multibump solution by -1, we obtain a solution with negative bumps at both ends, as well.

Observe that in the open-end case the domains Ω_R are contractible, and they are even convex if Γ is a segment. This means that to get multiplicity of sign changing solutions neither topological nor particular geometrical assumptions are needed. This stands in contrast with the case of positive solutions where it has been conjectured that for some power-type nonlinearities only one positive solution exists in any convex domain [7], as it does in a ball. Of course this difference between multiplicity of positive and sign changing solutions can be easily understood by looking at odd nonlinearities. In fact, if f is odd (for example, if $f(u) = |u|^{p-1}u$, $p \in (1, p_S)$) it is well known that infinitely many sign changing solutions exist in any bounded domain. Our results *do not assume that f is odd*, therefore multiplicity of sign changing solutions is not so obvious. In fact, if f is not odd only few multiplicity results are available, see e.g. [3, 6].

Dancer exhibited positive solutions with multiple bumps for "dumbbell shaped domains" [7, 8]. Sign changing solutions may also be constructed in domains of this type. On the other hand, if Γ is a segment, Theorem 1.2 yields examples of *convex* domains in which problem (1.2) has at least k nodal solutions with up to $k+1$ peaks, for any given k, without assuming that f is odd. We believe this is the first result of this type.

As in other similar problems, the procedure to prove Theorems 1.1 and 1.2 is to consider approximate solutions to problem (1.2) and then show that near them a true solution exists. So, to start, we need to make a good guess as to what the approximate solutions should be. The geometry of our expanding domains suggests looking at functions of the form

$$U^+_{x_{R,1},R} + U^-_{x_{R,2},R} + U^+_{x_{R,3},R} + U^-_{x_{R,4},R} + \cdots$$

for finitely many points $x_{R,1}, x_{R,2}, x_{R,3}, x_{R,4}, \ldots$, ordered along the curve, whose number is even if the curve is closed. Then some estimates are needed to show that these are indeed good approximate solutions and to compute the order of the approximation. To prove the existence of a true solution near them we follow a well-known Lyapunov-Schmidt reduction procedure, which relies on the contraction mapping principle. This requires again careful estimates on the approximate solutions and their linearization. Finally, a critical point of the reduced problem is obtained by a minimization. Here the crucial role is played by the fact that the interaction between a positive and a negative bump increases the value of the energy functional. This explains why the bumps should be placed along the tube with alternating signs and why the number of bumps must be even in the closed tube case (Theorem 1.1). In the open-end case (Theorem 1.2) the energy also increases as a bump approaches an end of the tube. Therefore, in both cases, a solution to the reduced problem is obtained by minimizing the energy.

It is harder to prove similar results when Γ is a higher dimensional manifold, instead of a curve. For positive solutions some results were obtained by Dancer and Yan [10] when Γ is the boundary of a convex domain. Positive multibump solutions in a tubular neighborhood of an expanding compact manifold have been constructed in [2]. The problem of constructing sign changing solutions in such domains is more subtle and requires minimax arguments.

The outline of the paper is as follows: In section 2 we have collected some tools, and results about the linear problem. Section 3 contains the essential energy estimates, while in section 4 we describe the finite dimensional reduction and prove our main results.

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2. Preliminaries

2.1. **Algebraic and geometric tools.** We start with some elementary lemmas which will be used later to estimate the interactions.

Lemma 2.1. *Suppose that* $\mu_k > \bar{\mu} \geq 0$ *for* $k = 1, 2, 3$. *Then there is* $C > 0$ *such that the inequalities*

$$\int_{\mathbb{R}^N} \mathrm{e}^{-\mu_1|x-x_1|}\mathrm{e}^{-\mu_2|x-x_2|}\,\mathrm{d}x \leq C\mathrm{e}^{-\bar{\mu}|x_1-x_2|} \tag{2.1}$$

and

$$\int_{\mathbb{R}^N} \mathrm{e}^{-\mu_1|x-x_1|}\mathrm{e}^{-\mu_2|x-x_2|}\mathrm{e}^{-\mu_3|x-x_3|}\,\mathrm{d}x \le C\exp\Big(-\bar\mu \min_{x\in\mathbb{R}^N}\sum_{k=1}^{3}|x-x_k|\Big) \tag{2.2}$$

hold true for all $x_1, x_2, x_3 \in \mathbb{R}^N$.

Proof. Since $\bar\mu|x_1-x_2|+(\mu_2-\bar\mu)|x-x_2| \le \bar\mu(|x-x_1|+|x-x_2|)+(\mu_2-\bar\mu)|x-x_2| \le \mu_1|x-x_1|+\mu_2|x-x_2|$, we have that

$$\int_{\mathbb{R}^N} \mathrm{e}^{-\mu_1|x-x_1|}\mathrm{e}^{-\mu_2|x-x_2|}\,\mathrm{d}x \le \int_{\mathbb{R}^N} \mathrm{e}^{-\bar\mu|x_1-x_2|}\mathrm{e}^{-(\mu_2-\bar\mu)|x-x_2|}\,\mathrm{d}x = C\mathrm{e}^{-\bar\mu|x_1-x_2|},$$

as claimed. The proof of the other inequality is similar. □

Lemma 2.2. *There exists* $\alpha \in (1/2, 1]$ *with the following property: for any given* $\widetilde{C}_1 \ge 1$ *and* $n \in \mathbb{N}$ *there is a constant* $\widetilde{C}_2 = \widetilde{C}_2(\alpha, n, \widetilde{C}_1) > 0$ *such that the inequalities*

$$\left| f\Big(\sum_{i=1}^{n} u_i\Big) - \sum_{i=1}^{n} f(u_i)\right| \le \widetilde{C}_2 \sum_{i<j}|u_iu_j|^{\alpha}, \tag{2.3}$$

$$\left| F\Big(\sum_{i=1}^{n} u_i\Big) - \sum_{i=1}^{n} F(u_i) - \sum_{i\ne j} f(u_i)u_j\right| \le \widetilde{C}_2\Big(\sum_{i<j}|u_iu_j|^{2\alpha} + \sum_{i<j<k}|u_iu_ju_k|^{2/3}\Big), \tag{2.4}$$

hold true for all $u_1, u_2, \ldots, u_n \in \mathbb{R}$ *with* $|u_i| \le \widetilde{C}_1$.

Proof. Observe that (H3) implies that there is a constant $C > 0$ such that

$$\left|f^{(k)}(u)\right| \le C\,|u|^{p_1-k} \qquad \text{if } |u| \le \widetilde{C}_1,\ u \ne 0. \tag{2.5}$$

Set $\alpha := \min\{(p_1+1)/4, 1\} \in (1/2, 1]$. It is tedious but elementary to prove that the inequalities

$$|f(u+v) - f(u) - f(v)| \le C\,|uv|^{\alpha} \tag{2.6}$$

and

$$|F(u+v) - F(u) - F(v) - f(u)v - f(v)u| \le C\,|uv|^{2\alpha} \tag{2.7}$$

hold true for some constant $C > 0$, if $|u|, |v| \le \widetilde{C}_1$. These are inequalities (2.3) and (2.4) for $n = 2$. For $n > 2$ inequalities (2.3) and (2.4) follow easily by induction on n. □

The right-hand side of inequality (2.4) indicates that we will need to consider triple interactions. The following lemma will be useful to estimate them.

Lemma 2.3. *Consider a triangle in* $\mathbb{R}^N$ *with vertices* $x_1, x_2, x_3 \in \mathbb{R}^N$ *and side lengths* $w \le v \le u$. *Denote* $s := \min_{x\in\mathbb{R}^N}\sum_{k=1}^{3}|x-x_k|$. *Then the following statements are true:*

(a) *If one of the interior angles is larger than or equal to* $2\pi/3$, *then* $s = v + w$.
(b) *In any case,* $s \ge (w + v + u)/2$.

Proof. The following facts from triangle geometry may be found in [13]. The minimum s is achieved at a unique point x_0 in $\mathbb{R}^N$. In case (a) that point is the vertex of the triangle with the largest interior angle, so the claim follows immediately.

To prove (b) observe that adding up the inequalities $|x_i - x_0| + |x_j - x_0| \geq |x_i - x_j|$, $i \neq j$, yields

$$2s = 2\sum_{k=1}^{3} |x_0 - x_k| \geq w + v + u \qquad \forall x \in \mathbb{R}^N,$$

as claimed. □

Lemma 2.4. *For $n \in \mathbb{N}$ there is a constant $C = C(n)$ such that if $x_1, x_2 \in \mathbb{R}^n$ satisfy $|x_1 - x_2| < 1$ and if $r \in [1, |x_1 - x_2| + 1]$ then*

$$\text{vol}_n\left(B_r(x_2) \smallsetminus B_1(x_1)\right) \leq C\left(|x_1 - x_2| + r - 1\right), \tag{2.8}$$

$$\sup_{x \in \partial B_r(x_2)} \text{dist}(x, \partial B_1(x_1)) \leq |x_1 - x_2| + r - 1, \tag{2.9}$$

$$\sup_{x \in \partial B_1(x_1)} \text{dist}(x, \partial B_r(x_2)) \leq |x_1 - x_2| + r - 1. \tag{2.10}$$

Here vol_n denotes the Lebesgue measure in $\mathbb{R}^n$ and $B_r(x) := \{y \in \mathbb{R}^n \mid |y - x| < r\}$.

Proof. Let ω_k denote the volume of the unit ball in $\mathbb{R}^k$. Set $d := |x_1 - x_2|$. Without loss of generality we may suppose that $x_1 = 0$ and $x_2 = (d, 0, \ldots, 0)$. Set $B_1 := B_1(0)$. Since $B_r(x_2) \smallsetminus B_1 \subset (B_1(x_2) \smallsetminus B_1) \cup (B_r(x_2) \smallsetminus B_1(x_2))$ and $r \in [1, 2]$, we have that

$$\begin{aligned}\text{vol}_n(B_r(x_2) \smallsetminus B_1) &\leq \text{vol}_n(B_1(x_2) \smallsetminus B_1) + \omega_n(r^n - 1) \\ &\leq \text{vol}_n(B_1 \smallsetminus B_1(x_2)) + \omega_n(2^n - 1)(r - 1).\end{aligned}$$

Write $x = (t, y) \in \mathbb{R}^n$ with $t \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$. By symmetry considerations,

$$\text{vol}_n(B_1 \smallsetminus B_1(x_2)) = \text{vol}_n\{x \in B_1 \mid |t| \leq d/2\} \leq \omega_{n-1} d.$$

Together with the previous inequality, this proves (2.8). An obvious geometric argument proves (2.9) and (2.10). □

2.2. **Analysis of linear operators and the limit problem.** Next we will show that $-\Delta + \lambda$ satisfies the strong maximum principle on $\mathbb{L}$ and Ω_R for R large if $\lambda > -\lambda_{1,1}$.

For $r > 0$ let $\lambda_{1,r}$ denote the smallest Dirichlet eigenvalue of $-\Delta$ in the open ball $B_r^{N-1} := \{\eta \in \mathbb{R}^{N-1} \mid |\eta| < r\}$ of radius r in $\mathbb{R}^{N-1}$, and let $\vartheta_{1,r}$ be the positive eigenfunction corresponding to $\lambda_{1,r}$, normalized by $\|\vartheta_{1,r}\|_{L^2} = 1$. The following result is well known.

Lemma 2.5. *If $\lambda_1(\mathbb{L})$ denotes the bottom of the spectrum of $-\Delta$ in $L^2(\mathbb{L})$ with Dirichlet boundary conditions, then $\lambda_1(\mathbb{L}) = \lambda_{1,1}$.*

Next, we construct a positive superharmonic function for $-\Delta + \lambda$ in Ω_R for R large. This allows to estimate the bottom of the spectrum of $-\Delta$ in $L^2(\Omega_R)$ from below and provides a maximum principle for $-\Delta + \lambda$.

As before, we write a point in $\mathbb{R}^N$ as (ξ, η), where $\xi \in \mathbb{R}$ and $\eta \in \mathbb{R}^{N-1}$.

Lemma 2.6. *If $\lambda > -\lambda_{1,1}$, there exists a superharmonic function for $-\Delta + \lambda$ in $C^2(\mathbb{L}) \cap C(\overline{\mathbb{L}})$ which is positive on $\overline{\mathbb{L}}$. If R is large enough then there exists a superharmonic function for $-\Delta + \lambda$ in $C^2(\Omega_R) \cap C(\overline{\Omega_R})$ which is positive on $\overline{\Omega_R}$.*

Proof. We fix $r > 1$ close enough to 1 so that $\lambda_{1,r} + \lambda > 0$. Then $W(\xi, \eta) := \vartheta_{1,r}(\eta)$ satisfies

$$(-\Delta + \lambda)W = (\lambda_{1,r} + \lambda)W > 0 \text{ in } \mathbb{L} \quad \text{and} \quad \min_{\mathbb{L}} W > 0.$$

This proves the first assertion.

To prove the second one note first that, for $R \geq 1$ large enough, the set $\Omega_{R,r} := \{x \in \mathbb{R}^N \,|\text{dist}(x, \Gamma_R) < r\}$ is a tubular neighborhood of Γ_R. Since $\vartheta_{1,r}$ is radial, we may write $\vartheta_{1,r}(\eta) = \vartheta_{1,r}(|\eta|)$ and define

$$W(x) := \vartheta_{1,r}(\text{dist}(x, \Gamma_R)) \quad \text{for } x \in \Omega_{R,r}.$$

Clearly, $\min_{\Omega_R} W > 0$ for R large enough. We claim that

$$W \in C^2(\Omega_R) \cap C(\overline{\Omega_R}) \tag{2.11}$$

and

$$\min_{\Omega_R} ((-\Delta + \lambda)W) > 0 \tag{2.12}$$

for R large enough. To prove this claims we fix $y_0 \in \Omega_R$ and we define locally, around y_0, a diffeomorphism from Ω_R to the unit normal bundle of Γ_R as follows: after a change of coordinates we may assume that $0 \in \Gamma_R$ and that $\text{dist}(y_0, \Gamma_R) = |y_0|$. We may also assume that the tangent space to Γ_R at 0 is $\mathbb{R} \times \{0\}$. Then, $y_0 \in \{0\} \times \mathbb{R}^{N-1}$. Let $\tau\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^N$ be a parametrization by arc length of Γ such that $\tau(0) = 0$ and $\tau'(0) = (1, 0)$. For $\xi \in (-R\varepsilon, R\varepsilon)$ and $\eta \in \mathbb{R}^{N-1}$, set $\tau_R(\xi) := R\tau(\frac{\xi}{R})$ and let $h_R(\xi, \eta)$ be the orthogonal projection of $(0, \eta)$ onto the space $\tau'(\xi)^\perp = \{x \in \mathbb{R}^N : x \cdot \tau'(\frac{\xi}{R}) = 0\}$. Now define

$$\Phi_R(\xi, \eta) := \tau_R(\xi) + \frac{|\eta|}{|h_R(\xi, \eta)|} h_R(\xi, \eta).$$

Note that $\Phi_R(0, \eta) = (0, \eta)$. Moreover,

$$\mathrm{D}\Phi_R(0, \eta) = \begin{pmatrix} 1 - \frac{1}{R}\left[(0, \eta) \cdot \tau''(0)\right] & 0 \\ 0 & I_{N-1} \end{pmatrix}. \tag{2.13}$$

Therefore, Φ_R is a C^2-diffeomorphism between neighborhoods of $\overline{\{0\} \times B_1^{N-1}}$ for R large enough. Note that, since $h_R(\xi, \eta)$ is orthogonal to Γ_R at $\tau_R(\xi)$,

$$\text{dist}(\Phi_R(\xi, \eta), \Gamma_R) = \left|\Phi_R(\xi, \eta) - \frac{1}{R}\tau(\xi)\right| = |\eta|.$$

This implies that

$$W(\Phi_R(\xi, \eta)) = \vartheta_{1,r}(|\eta|).$$

So, since Φ_R is a local C^2-diffeomorphism at y_0, this identity proves (2.11).

To prove (2.12) it is enough to show that

$$(-\Delta + \lambda)W(0, \eta) \geq C > 0 \tag{2.14}$$

for $\eta \in B_1^{N-1}$ and large R, where C is independent of y_0 and R. A straightforward computation shows that

$$(-\Delta + \lambda)W(0, \eta) = (\lambda_{1,r} + \lambda)\, \vartheta_{1,r}(\eta) + O(\left|\mathrm{D}^2\Phi_R(0, \eta)\right|),$$

independently of y_0, and that that

$$\mathrm{D}^2\Phi_R(0, \eta) \to 0 \qquad \text{as } R \to \infty, \text{ independently of } y_0 \text{ and } \eta. \tag{2.15}$$

Since $\vartheta_{1,r}$ is positive and continuous on $\overline{B_1^{N-1}}$ we may set

$$C := \frac{\lambda_{1,r} + \lambda}{2} \min_{|\eta| \le 1} \vartheta_{1,r}(\eta) > 0$$

and obtain (2.14). □

Corollary 2.7. *If $\lambda_1(\Omega_R)$ denotes the bottom of the spectrum of $-\Delta$ in $L^2(\Omega_R)$ with Dirichlet boundary conditions, then*

$$\liminf_{R\to\infty} \lambda_1(\Omega_R) \ge \lambda_{1,1}.$$

Proof. A standard argument, using Lemma 2.6, proves this claim. □

The following fact will play a crucial role to obtain asymptotic estimates for the energy functional and its gradient.

Corollary 2.8. *If $\lambda > -\lambda_{1,1}$ the operator $-\Delta + \lambda$ satisfies the strong maximum principle in any subdomain of $\mathbb{L}$ and in any subdomain of Ω_R for R large enough.*

Proof. This follows from Lemma 2.6 and [14, Theorem 1]. □

We shall also need the following decay estimates for the solutions $U^{\pm}$ to the limit problem (1.4). They follow immediately from [5, Proposition 4.2].

Lemma 2.9. *There are constants $C_1, C_2 > 0$ such that*

$$C_1 \mathrm{e}^{-\mu|\xi|} \vartheta_{1,1}(\eta) \le |U^{\pm}(\xi,\eta)| \le C_2 \mathrm{e}^{-\mu|\xi|} \vartheta_{1,1}(\eta) \qquad \text{for all } (\xi,\eta) \in \mathbb{L}$$

where $\mu := \sqrt{\lambda + \lambda_{1,1}}$.

3. Asymptotics of the energy and its gradient

We assume from now on that $\lambda > -\lambda_{1,1}$. Let $\mathbb{L}_s := \{(\xi,\eta) \in \mathbb{R}^1 \times \mathbb{R}^{N-1} : |\eta| < s\}$. We fix $r_0 > 1$ such that $\lambda_{1,r_0} + \lambda > 0$ and, for $R > 0$, $x \in \Gamma_R$ and $s \in [1, r_0]$, we set

$$\mathbb{L}_{s,x} := \{x + A_x^{-1}(z) : z \in \mathbb{L}_s\}$$

with A_x as in (1.5). Note that the first eigenvalue of $-\Delta$ in $H_0^1(\Omega_R \cap \mathbb{L}_{s,x})$ satisfies $\lambda_1(\Omega_R \cap \mathbb{L}_{s,x}) + \lambda > 0$ for large R, because $\Omega_R \cap \mathbb{L}_{s,x}$ is an open bounded subset of $\mathbb{L}_{r_0,x}$. We write $V^{\pm}_{x,s,R}$ for the unique solution to the problem

$$\text{(3.1)} \qquad \begin{cases} -\Delta u + \lambda u = f(U^{\pm}_{x,R}) & \text{in } \Omega_R \cap \mathbb{L}_{s,x}, \\ u = 0 & \text{on } \partial\left(\Omega_R \cap \mathbb{L}_{s,x}\right), \end{cases}$$

with $U^{\pm}_{x,R}$ as in (1.5). By the maximum principle and assuption (H4), $V^{+}_{x,s,R}$ is positive and $V^{-}_{x,s,R}$ is negative for large R. We extend $V^{\pm}_{x,s,R}$ to all of $\mathbb{R}^N$ by defining it as 0 outside of $\Omega_R \cap \mathbb{L}_{s,x}$. When $s = 1$ we omit it from the notation and write $\mathbb{L}_x$, $V_{x,R}$ instead of $\mathbb{L}_{1,x}$, $V_{x,1,R}$.

3.1. **The closed tube case.** In this subsection we assume that $\gamma(0) = \gamma(1)$. The following decay estimates hold true.

Lemma 3.1. *For each $s \in [1, r_0)$ there are positive constants c_3, c_4 and R_0, independent of $x \in \Gamma_R$, such that all quantities*

$$\left|U^{\pm}_{x,R}(y)\right|, \quad \left|\nabla U^{\pm}_{x,R}(y)\right|, \quad \left|V^{\pm}_{x,s,R}(y)\right|, \quad \left|\nabla V^{\pm}_{x,s,R}(y)\right|,$$

are bounded by $c_3 e^{-c_4|y-x|}$ for all $R \geq R_0$ and almost all $y \in \mathbb{R}^N$. Moreover,

$$\left|D^2 U^{\pm}_{x,R}(y)\right| \quad \text{and} \quad \left|D^2 V^{\pm}_{x,s,R}(y)\right|$$

are bounded uniformly in $\mathbb{L}_x$ and $\Omega_R \cap \mathbb{L}_{s,x}$ respectively, independently of $R \geq R_0$.

Proof. Lemma 2.9, together with standard regularity estimates, yields the estimates for $U^{\pm}_{x,R}$ and its derivatives.
To prove the estimates for $V^{\pm}_{x,s,R}$ we assume without loss of generality that $x = 0$ and that $\mathbb{R} \times \{0\}$ is the tangent space to Γ_R at 0. Then there exists $\tilde{c}_s > 0$ such that $\vartheta_{1,r_0}(\eta) \geq \tilde{c}_s$ for all $\eta \in B^{N-1}_s$, where ϑ_{1,r_0} is the positive first Dirichlet eigenfunction of $-\Delta$ in the ball of radius r_0 (as in the beginning of subsection 2.2). We write $y \in \mathbb{L}_s$ as (ξ, η) with $\xi \in \mathbb{R}$ and $\eta \in B^{N-1}_s$, and set

$$W(y) := e^{-\nu|\xi|}\vartheta_{1,r_0}(\eta)$$

where ν is a small positive constant, independent of R, which will be fixed next. A straightforward computation gives

$$\begin{aligned} -\Delta W(y) + \lambda W(y) &= \left(\frac{(N-1)\,\nu}{|\xi|} - \nu^2 + \lambda_{1,r_0} + \lambda\right) W(y) \\ &> \left(\lambda_{1,r_0} + \lambda - \nu^2\right) \tilde{c}_s e^{-\nu|\xi|}. \end{aligned}$$

Since $\lambda_{1,r_0} + \lambda > 0$ we have that $\lambda_{1,r_0} + \lambda - \nu^2 > 0$ if ν is small enough. On the other hand, assumption (H3) on f together with Lemma 2.9 yield that

$$f(U^{+}_{x,R}) \leq b_1 e^{-\mu p_1 |\xi|},$$

for some large enough $b_1 > 0$. Since $V^{+}_{x,s,R}$ satisfies (3.1) the maximum principle implies that $V^{+}_{x,s,R} \leq b_2 W$ with $b_2 := b_1 \tilde{c}_s^{-1}\left(\lambda_{1,r_0} + \lambda - \nu^2\right)^{-1}$. This gives the exponential bound on $V^{+}_{x,s,R}$. Similarly for $V^{-}_{x,s,R}$. Regularity estimates, using the results in [11], yield the estimates for its derivatives. □

Set

$$F(u) := \int_0^u f(s)\, \mathrm{d}s \qquad \text{if } u \in \mathbb{R}.$$

Then, by (H3),

$$|F(u)| \leq C(|u|^{p_1+1} + |u|^{p_2+1}) \qquad \text{for all } u \in \mathbb{R}. \tag{3.2}$$

Lemma 3.2. *For $s \in [1, r_0)$ and $p \in (0, \infty)$ the asymptotic estimates*

$$\int_{\mathbb{R}^N} |V^{\pm}_{x,s,R} - U^{\pm}_{x,R}|^p = O(R^{-\min\{p,1\}}), \tag{3.3}$$

$$\int_{\mathbb{R}^N} |\nabla V^{\pm}_{x,s,R} - \nabla U^{\pm}_{x,R}|^2 = O(R^{-1}), \tag{3.4}$$

$$\int_{\mathbb{R}^N} |F(V^{\pm}_{x,s,R}) - F(U^{\pm}_{x,R})| = O(R^{-1}), \tag{3.5}$$

$$\int_{\mathbb{R}^N} |f(V^{\pm}_{x,s,R}) - f(U^{\pm}_{x,R})|^p = O(R^{-\min\{p,1\}}), \tag{3.6}$$

hold true as $R \to \infty$, independently of $x \in \Gamma_R$.

Proof. Let x be a point on Γ. After translation and rotation we may assume that $x = 0$ and that $\mathbb{R} \times \{0\}$ is the tangent space to Γ at 0. Since Γ is compact there exist $\delta, \rho > 0$, independent of x, and a C^3-function $h : (-\rho, \rho) \to B^{N-1}_\delta$ such that

$$\Gamma \cap \big((-\rho, \rho) \times B^{N-1}_\delta\big) = \{(\xi, h(\xi)) \mid \xi \in (-\rho, \rho)\},$$

and the derivatives of h up to the order 3 are bounded independently of $\xi \in (-\rho, \rho)$ and $x \in \Gamma$. Setting $h_R(\xi) := Rh(\xi/R)$ we have that

$$\widetilde{\Gamma}_R := \Gamma_R \cap \big((-\rho R, \rho R) \times B^{N-1}_{\delta R}\big) = \{(\xi, h_R(\xi)) \mid \xi \in (-\rho R, \rho R)\}.$$

An easy argument using Taylor's theorem and geometric considerations shows that there is a constant C, independent of x, such that

$$|h_R(\xi)| \le \frac{C\xi^2}{R}, \quad |h'_R(\xi)| \le \frac{C\,|\xi|}{R} \quad \text{and} \quad |y - h_R(\xi)| \le 1 + \frac{C(1+\xi^2)}{R^2} \tag{3.7}$$

for all $\xi \in (-\rho R + 1, \rho R - 1)$ and $y \in \mathbb{R}^{N-1}$ with $(\xi, y) \in \Omega_R$. It follows that

$$\{\xi\} \times B^{N-1}_1(h_R(\xi)) \subset \big[\{\xi\} \times \mathbb{R}^{N-1}\big] \cap \Omega_R \subset \{\xi\} \times B^{N-1}_{1+C(1+|\xi|^2)/R^2}(h_R(\xi)) \tag{3.8}$$

for all $\xi \in (-\rho R + 1, \rho R - 1)$ and R large enough. Consider the set

$$Q_R := (-R^{1/4}, R^{1/4}) \times B^{N-1}_s \subset \mathbb{L}_s.$$

We express $\mathbb{R}^N$ as the union of the sets

$$\mathbb{R}^N \smallsetminus Q_R, \qquad Q_R \cap (\Omega_R \smallsetminus \mathbb{L}), \qquad Q_R \cap (\mathbb{L} \smallsetminus \Omega_R), \qquad Q_R \cap \mathbb{L} \cap \Omega_R. \tag{3.9}$$

We will show that the estimates (3.3), (3.4), (3.5), (3.6), hold true for the integrals over each one of these sets. Note that the integrals over $Q_R \smallsetminus (\mathbb{L} \cup \Omega_R)$ are zero.

Claim 1. *Estimate* (3.3) *holds true for the integral over $\mathbb{R}^N \smallsetminus Q_R$.*

By Lemma 3.1 there are positive constants $\widetilde{C}_1, \widetilde{C}_2$ such that

$$|V^{\pm}_{x,s,R}(\xi, \eta) - U^{\pm}_{x,R}(\xi, \eta)|^p \le \widetilde{C}_1 \mathrm{e}^{-\widetilde{C}_2(|\xi|+|\eta|)} \tag{3.10}$$

for all $(\xi, \eta) \in \mathbb{R} \times \mathbb{R}^{N-1}$. This immediately yields Claim 1.

Claim 2. *Estimate* (3.3) *holds true for the integral over $Q_R \cap (\Omega_R \smallsetminus \mathbb{L})$.*

By (3.10), (3.8), Lemma 2.4 and (3.7) it holds that

$$\begin{aligned}
&\int_{Q_R\cap(\Omega_R\setminus\mathbb{L})}|V^{\pm}_{x,s,R}-U^{\pm}_{x,R}|^p\\
&\le \widetilde{C}_1\int_{-R^{1/4}}^{R^{1/4}} \mathrm{e}^{-\widetilde{C}_2|\xi|}\mathrm{vol}_{N-1}\left(B^{N-1}_{1+C(1+|\xi|^2)/R^2}(h_R(\xi))\setminus B^{N-1}_1(0)\right)\mathrm{d}\xi\\
&\le C\int_{-R^{1/4}}^{R^{1/4}} \mathrm{e}^{-\widetilde{C}_2|\xi|}(|h_R(\xi)|+C(1+\xi^2)/R^2)\;\mathrm{d}\xi\\
&\le \frac{C}{R}\int_{-\infty}^{\infty} \mathrm{e}^{-\widetilde{C}_2|\xi|}(1+\xi^2)\;\mathrm{d}\xi = O(R^{-1})
\end{aligned}$$

as $R\to\infty$.

Claim 3. *Estimate* (3.3) *holds true for the integral over* $Q_R\cap(\mathbb{L}\setminus\Omega_R)$.

The proof is similar to that of Claim 2, using this time the first inclusion in (3.8).

Claim 4. *Estimate* (3.3) *holds true for the integral over* $Q_R\cap\mathbb{L}\cap\Omega_R$.

Set $D_R := Q_R\cap\mathbb{L}\cap\Omega_R$. First we prove that, for some suitable constant C independent of x and R,

$$\left|V^{\pm}_{x,s,R}(\xi,\eta)-U^{\pm}_{x,R}(\xi,\eta)\right|\le C\mathrm{e}^{-C_4|\xi|}\frac{1+\xi^2}{R} \tag{3.11}$$

for all $(\xi,\eta)\in\partial D_R$. Let $(\xi,\eta)\in\partial D_R$. If $(\xi,\eta)\in\partial\mathbb{L}$, Lemma 2.4, together with (3.8), and (3.7), yields

$$\mathrm{dist}((\xi,\eta),\partial\Omega_R)\le|h_R(\xi)|+C\frac{1+\xi^2}{R^2}\le C\frac{1+\xi^2}{R}. \tag{3.12}$$

Similarly, if $(\xi,\eta)\in\partial\Omega_R$ then

$$\mathrm{dist}((\xi,\eta),\partial\mathbb{L})\le C\frac{1+\xi^2}{R}.$$

Since $U^{\pm}_{x,R}$ vanishes on $\partial\mathbb{L}$ and $V^{\pm}_{x,s,R}$ vanishes on $\partial\Omega_R$, the estimates in Lemma 3.1 yield inequality (3.11). Next we set $W(y):=e^{-\nu|\xi|}\vartheta_{1,r_0}(\eta)$ with $\nu\in(0,C_4)$ as in the proof of Lemma 3.1. By (3.11) there exists $C>0$ such that

$$\left|V^{\pm}_{x,s,R}(\xi,\eta)-U^{\pm}_{x,R}(\xi,\eta)\right|\le\frac{C}{R}W(\xi,\eta)$$

for all $(\xi,\eta)\in\partial D_R$. Since $V^{\pm}_{x,R}-U^{\pm}_{x,R}$ is harmonic for $-\Delta+\lambda$ in D_R the maximum principle implies that

$$\left|V^{\pm}_{x,s,R}(\xi,\eta)-U^{\pm}_{x,R}(\xi,\eta)\right|\le\frac{C}{R}W(\xi,\eta)=\frac{C}{R}e^{-\nu|\xi|}\vartheta_{1,r_0}(\eta)$$

for all $(\xi,\eta)\in D_R$, with C independent of x and R. Therefore,

$$\int_{D_R}\left|V^{\pm}_{x,s,R}-U^{\pm}_{x,R}\right|^p=O(R^{-p}) \tag{3.13}$$

as $R\to\infty$. This proves Claim 4.

Claim 5. *Estimate* (3.4) *holds true for the integrals over* $\mathbb{R}^N\setminus Q_R$, $Q_R\cap(\Omega_R\setminus\mathbb{L})$ *and* $Q_R\cap(\mathbb{L}\setminus\Omega_R)$.

The same arguments as in the proofs of Claims 1, 2 and 3 yield this claim.

Claim 6. *Estimate* (3.4) *holds true for the integral over* $Q_R \cap \mathbb{L} \cap \Omega_R$.

Set $D_R := Q_R \cap \mathbb{L} \cap \Omega_R$. The functions $U^{\pm}_{x,R}$ and $V^{\pm}_{x,R}$ can be extended to C^2-functions in neighborhoods of $\mathbb{L}$ and Ω_R, respectively. Denote by Y_R the difference of these extensions on a neighborhood of $\overline{D_R}$. Note that D_R has Lipschitz boundary if R is large enough. Hence we can apply the Gauss-Green theorem (see e.g. [15, Theorem 5.8.2] and the remark following it) and obtain that

$$\int_{D_R} |\nabla Y_R|^2 = \int_{\partial D_R} Y_R\, n_R(x)\cdot \nabla Y_R \,\mathrm{d}H_{N-1}(x) - \lambda \int_{D_R} Y_R^2 \tag{3.14}$$

Here $n_R(x)$ denotes the measure theoretic exterior normal to ∂D_R at x, and H_{N-1} denotes $(N-1)$-dimensional Hausdorff measure. By Lemma 3.1, ∇Y_R is bounded uniformly and independently of R. Hence (3.14) and (3.11) imply

$$\begin{aligned}\int_{D_R} |\nabla Y_R|^2 &\le \frac{C}{R}\left(\int_{\partial D_R} \mathrm{e}^{-C_4|\xi|}(1+|\xi|^2)\,\mathrm{d}H_{N-1}(x) + \int_{D_R} \mathrm{e}^{-C_4|\xi|}(1+|\xi|^2)\,\mathrm{d}\xi\,\mathrm{d}\eta\right)\\ &= O(R^{-1}).\end{aligned}$$

This proves Claim 6.

Claim 7. *Estimates* (3.5) *and* (3.6) *hold true.*

These estimates follow easily from (3.3) since $U^{\pm}_{x,R}$ and $V^{\pm}_{x,R}$ are bounded uniformly as $R \to \infty$ and F and f are continuously differentiable. □

The energy functional for the Dirichlet problem $-\Delta u + \lambda u = f(u)$ in a domain $\Omega \subseteq \mathbb{R}^N$ is given by

$$J_\Omega(u) := \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2) - \int_\Omega F(u), \qquad u \in H^1_0(\Omega).$$

By (H2), (H3) and (3.2) J_Ω is well defined and twice continuously differentiable on $H^1_0(\Omega)$, with $\mathrm{D}^2 J_\Omega$ globally Hölder continuous on bounded subsets of $H^1_0(\Omega)$.

Lemma 3.3. *The estimates*

$$\sup_{x\in\Gamma_R} \|V^{\pm}_{x,R} - U^{\pm}_{x,R}\|_{H^1(\mathbb{R}^N)} = O(R^{-1/2}), \tag{3.15}$$

$$\sup_{x\in\Gamma_R} |J_{\Omega_R}(V^{\pm}_{x,R}) - J_{\mathbb{L}}(U^{\pm})| = O(R^{-1}), \tag{3.16}$$

$$\sup_{x\in\Gamma_R} \|\nabla J_{\Omega_R}(V^{\pm}_{x,R})\|_{H^1_0(\Omega_R)} = O(R^{-1/2}), \tag{3.17}$$

hold true as $R \to \infty$.

Proof. Estimates (3.15) and (3.16) follow immediately from Lemma 3.2. To prove the third one we choose $s \in (1, r_0)$ and a cut-off function $\chi \in C^\infty(\mathbb{R}^{N-1})$ with $\chi(\eta) = 1$ if $|\eta| \le 1$ and $\chi(\eta) = 0$ if $|\eta| \ge s$. Fix R and $x \in \Gamma_R$. Assuming that $x = 0$ and that $\mathbb{R}\times\{0\}$ is the tangent space to Γ_R at 0, we write $v \in H^1_0(\Omega_R)$ as $v = v_1 + v_2$ where $v_1(\xi,\eta) := \chi(\eta)v(\xi,\eta)$. Then $v_1 \in H^1_0(\Omega_R \cap \mathbb{L}_{s,x})$, $\mathrm{supp}(v_2) \subset \Omega_R \smallsetminus \mathbb{L}_x$ and there exists a constant c_s, independent of R and x, such that $\|v_1\|_{H^1(\mathbb{R}^N)} \le c_s \|v\|_{H^1(\mathbb{R}^N)}$ for all $v \in H^1_0(\Omega_R)$. From the definition of $V^{\pm}_{x,s,R}$ and Lemma 3.2 we obtain

$$\begin{aligned}\left|DJ_R(V^{\pm}_{x,R})v\right| &= \left|DJ_R(V^{\pm}_{x,R})v_1\right| \\ &\le \left|DJ_R(V^{\pm}_{x,s,R})v_1\right| + \left|DJ_R(V^{\pm}_{x,R})v_1 - DJ_R(V^{\pm}_{x,s,R})v_1\right| \\ &\le \left|\int_{\mathbb{R}^N}\Big(f(U^{\pm}_{x,R}) - f(V^{\pm}_{x,s,R})\Big)v_1\right| + O(R^{-1/2})\left\|v_1\right\|_{H^1(\mathbb{R}^N)} \\ &\le O(R^{-1/2})\left\|v\right\|_{H^1(\mathbb{R}^N)},\end{aligned}$$

as claimed. □

For $m = 1, 2$ we consider functions $g_m : \mathbb{R}^+ \to \mathbb{R}^+$ (to be fixed later) satisfying

$$g_2 < g_1, \tag{3.18}$$

$$g_m(R) \to \infty \qquad \text{as } R \to \infty, \text{ for } m = 1, 2, \tag{3.19}$$

$$g_m(R) = o(R) \qquad \text{as } R \to \infty, \text{ for } m = 1, 2. \tag{3.20}$$

Let $D_{m,R}$ be the set of points $(x_1, x_2, \ldots, x_n)$ in $(\Gamma_R)^n$ such that there exist $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$ and $|x_i - x_j| \le g_m(R)$, and let

$$\mathcal{U}_{m,R} := \{(x_1, x_2, \ldots, x_n) \in (\Gamma_R)^n \smallsetminus D_{m,R} \mid (x_1, x_2, \ldots, x_n) \text{ is an } n\text{-chain}\}, \tag{3.21}$$

see (1.6) for the definition of an n-chain. Then $\mathcal{U}_{1,R}$ and $\mathcal{U}_{2,R}$ are open subsets of $(\Gamma_R)^n$ such that $\overline{\mathcal{U}_{1,R}} \subset \mathcal{U}_{2,R}$. For $i, j \in \{1, 2, \ldots, n\}$ we set

$$d_n(i, j) := \min\{|i - j|, |i - j + n|, |i - j - n|\}.$$

$d_n(i, j)$ is the distance from i to the set of integers which are congruent to j mod n.

Lemma 3.4. *For R large enough and every $(x_1, x_2, \ldots, x_n) \in \overline{\mathcal{U}_{1,R}}$ we have that*

$$s(R) := \min_{x \in \mathbb{R}^N}\left(|x - x_i| + |x - x_j| + |x - x_\ell|\right) \ge 2g_1(R) \tag{3.22}$$

for any $i, j, \ell \in \{1, 2, \ldots, n\}$, and

$$|x_i - x_j| \ge \frac{4}{3}g_1(R) \qquad \text{if } d_n(i, j) \ge 2. \tag{3.23}$$

Proof. Since Γ is compact there exists $\varrho > 0$ with the following properties:

(i) If $x, y \in \Gamma$ and $0 < |x - y| < 2\varrho$ then there exists a connected component $\mathcal{C}$ of $\Gamma \smallsetminus \{x, y\}$ such that $|x - z| + |z - y| \le \frac{3}{2}|x - y|$ for every $z \in \mathcal{C}$.

(ii) If x, y, z are three different points in Γ, $|x - y| < 2\varrho$ and $|z - y| < 2\varrho$ then one of the angles of the triangle with vertices x, y, z is larger that $2\pi/3$.

Fix R large enough so that $\frac{g_1(R)}{R} < \varrho$.

Let $(x_1, x_2, \ldots, x_n) \in \overline{\mathcal{U}_{1,R}}$. If $\max\left(|x_j - x_i| + |x_\ell - x_j| + |x_i - x_\ell|\right) < 2\varrho R$, the points $\frac{x_i}{R}, \frac{x_j}{R}, \frac{x_\ell}{R} \in \Gamma$ satisfy the hypothesis of (ii) and, therefore, the triangle with vertices x_i, x_j, x_ℓ has an angle which is larger that $2\pi/3$. It follows from Lemma 2.3 that $s(R) \ge 2g_1(R)$. If, on the other hand, $|x_j - x_i| \ge 2\varrho R$ then $|x_j - x_i| \ge 2g_1(R)$, and Lemma 2.3 implies that $s(R) \ge 2g_1(R)$. This proves (3.22).

To prove (3.23) we argue by contradiction. Assume there are $(x_1, x_2, \ldots, x_n) \in \overline{\mathcal{U}_{1,R}}$ and $i, j \in \{1, 2, \ldots, n\}$ such that $d_n(i, j) \ge 2$ and $|x_i - x_j| < \frac{4}{3}g_1(R)$. Then $0 < \left|\frac{x_i}{R} - \frac{x_j}{R}\right| < 2\varrho$. Since $d_n(i, j) \ge 2$ there is a point x_ℓ in the n-chain, which lies between x_i and x_j, such that $\frac{x_\ell}{R}$ belongs to the connected component of $\Gamma \smallsetminus \{\frac{x_i}{R}, \frac{x_j}{R}\}$

to which the conclusion of (i) applies. Then, $2g_1(R) \le |x_i - x_\ell| + |x_\ell - x_j| \le \frac{3}{2}|x_i - x_j|$, a contradiction. □

In the rest of this subsection we assume that $n = 2k$. For $X = (x_1, x_2, \dots, x_n) \in \mathcal{U}_{2,R}$ we define $\varphi_R\colon \mathcal{U}_{2,R} \to H_0^1(\Omega_R)$ by

$$\varphi_R(X) := \sum_{i=1}^{k} (V^+_{x_{2i-1},R} + V^-_{x_{2i},R}). \tag{3.24}$$

For fixed $X = (x_1, x_2, \dots, x_n)$ it will be convenient to write

$$\overline{U}_i := \begin{cases} U^+_{x_i,R} & \text{if } i \text{ is odd,} \\ U^-_{x_i,R} & \text{if } i \text{ is even,} \end{cases} \qquad \overline{V}_i := \begin{cases} V^+_{x_i,R} & \text{if } i \text{ is odd,} \\ V^-_{x_i,R} & \text{if } i \text{ is even.} \end{cases} \tag{3.25}$$

Then

$$\varphi_R(X) = \sum_{i=1}^{n} \overline{V}_i.$$

Proposition 3.5. *Let α be as in* Lemma 2.2 *and fix $\alpha' \in (1/2, \alpha)$. Then*

$$\sup_{X \in \mathcal{U}_{2,R}} \|\nabla J_{\Omega_R}(\varphi_R(X))\|_{H_0^1(\Omega_R)} = O(\mathrm{e}^{-\alpha' \mu g_2(R)}) + O(R^{-1/2})$$

as $R \to \infty$.

Proof. Fix $X = (x_1, x_2, \dots, x_n) \in \mathcal{U}_{2,R}$. If $v \in H_0^1(\Omega_R)$ satisfies $\|v\|_{H_0^1(\Omega_R)} = 1$ then, using Lemmas 3.3, 2.2, 3.2, 3.1 and 2.1 we obtain

$$\begin{aligned}
|DJ_{\Omega_R}(\varphi_R(X))\,[v]| &= \left| \sum_{i=1}^{n} DJ_{\Omega_R}(\varphi_R(\overline{V}_i))\,[v] + \int_{\Omega_R} \left(\sum_{i=1}^{n} f(\overline{V}_i) - f\left(\sum_{i=1}^{n} \overline{V}_i \right) \right) v \right| \\
&\le \sum_{i=1}^{n} \|\nabla J_{\Omega_R}(\overline{V}_i)\|_{H_0^1(\Omega_R)} + \left(\int_{\Omega_R} \left| \sum_{i=1}^{n} f(\overline{V}_i) - f\left(\sum_{i=1}^{n} \overline{V}_i \right) \right|^2 \right)^{1/2} \\
&\le O(R^{-1/2}) + C \textstyle\sum_{i<j} \left(\displaystyle\int_{\Omega_R} |\overline{V}_i \overline{V}_j|^{2\alpha} \right)^{1/2} \\
&= O(R^{-1/2}) + C \textstyle\sum_{i<j} \left(\displaystyle\int_{\Omega_R} |\overline{U}_i \overline{U}_j|^{2\alpha} \right)^{1/2} \\
&= O(R^{-1/2}) + O(e^{-\alpha' \mu g_2(R)}).
\end{aligned}$$

These estimates are independent of the choice of X. □

Recall that $n = 2k$ and set

$$E_n := k\left[J_{\mathbb{L}}(U^+) + J_{\mathbb{L}}(U^-) \right].$$

Proposition 3.6. *There exists $\beta > 0$ such that*

$$\inf_{X \in \partial\mathcal{U}_{1,R}} J_{\Omega_R}(\varphi_R(X)) \ge E_n + \beta \mathrm{e}^{-\mu g_1(R)} + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3})$$

as $R \to \infty$.

Proof. If $X = (x_1, x_2, \dots, x_n) \in \partial\mathcal{U}_{1,R}$ there are $i_0, j_0 \in \{1, 2, \dots, n\}$ such that $|x_{i_0} - x_{j_0}| = g_1(R)$. By Lemma 3.4, $d_n(i_0, j_0) = 1$ for R large enough. Then, assumption (H4) implies that

$$f(\overline{U}_{i_0})\overline{U}_{j_0} \le 0. \tag{3.26}$$

On the other hand, it follows from Lemma 2.9 and property (3.20) that there exist $r, \varepsilon > 0$ such that $\left|f(\overline{U}_{i_0})\right| \geq \varepsilon$ and $\left|\overline{U}_{j_0}\right| \geq C_1 \mathrm{e}^{-\mu g_1(R)}$ in $B_r(x_{i_0})$ for R large enough, independently of the choice of $X \in \partial\mathcal{U}_{1,R}$. Hence

$$\frac{1}{2}\int_{\mathbb{R}^N} \left|f(\overline{U}_{i_0})\overline{U}_{j_0}\right| \,\mathrm{d}x \geq \beta \mathrm{e}^{-\mu g_1(R)} \tag{3.27}$$

for some $\beta > 0$ and large R. Moreover, Lemmas 3.4, 2.1 and 2.9 yield

$$\int_{\mathbb{R}^N} \left|f(\overline{U}_i)\overline{U}_j\right| \,\mathrm{d}x = o(\mathrm{e}^{-\mu g_1(R)}) \qquad \text{if } d_n(i,j) \geq 2, \tag{3.28}$$

as $R \to \infty$.

Since $\overline{U}_i$ and $\overline{V}_i$ are uniformly bounded, using Lemma 2.2, estimate (3.3), and Lemmas 3.4 and 2.1, we obtain

$$\begin{aligned} &\left|\int_{\Omega_R}\left[F(\sum_i \overline{V}_i) - \sum_i F(\overline{V}_i)\right] - \sum_{i\neq j}\int_{\Omega_R} f(\overline{V}_i)\overline{V}_j\right| \\ &\leq C\sum_{i<j}\int_{\Omega_R}\left|\overline{V}_i\overline{V}_j\right|^{2\alpha} + C\sum_{i<j<k}\int_{\Omega_R}\left|\overline{V}_i\overline{V}_j\overline{V}_k\right|^{2/3} \\ &= C\sum_{i<j}\int_{\Omega_R}\left|\overline{U}_i\overline{U}_j\right|^{2\alpha} + C\sum_{i<j<k}\int_{\Omega_R}\left|\overline{U}_i\overline{U}_j\overline{U}_k\right|^{2/3} + O(R^{-2/3}) \\ &= o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}). \end{aligned} \tag{3.29}$$

Therefore, using estimates (3.16), (3.3), (3.26), (3.27) and (3.28) we conclude that

$$\begin{aligned} J_{\Omega_R}(\varphi_R(X)) =& \sum_{i=1}^n J_{\Omega_R}(\overline{V}_i) + \frac{1}{2}\sum_{i\neq j}\int_{\Omega_R}\left(\nabla\overline{V}_i\cdot\nabla\overline{V}_j + \lambda\overline{V}_i\overline{V}_j\right)\mathrm{d}x \\ & - \int_{\Omega_R}\left[F(\sum_i\overline{V}_i) - \sum_i F(\overline{V}_i)\right]\mathrm{d}x \\ =& E_n + \frac{1}{2}\sum_{i\neq j}\int_{\Omega_R} f(\overline{U}_i)\overline{V}_j\,\mathrm{d}x - \sum_{i\neq j}\int_{\Omega_R} f(\overline{V}_i)\overline{V}_j\,\mathrm{d}x \\ & + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}) \\ =& E_n - \frac{1}{2}\sum_{i\neq j}\int_{\Omega_R} f(\overline{U}_i)\overline{U}_j\,\mathrm{d}x + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}) \\ \geq& E_n + \frac{1}{2}\int_{\mathbb{R}^N}\left|f(\overline{U}_{i_0})\overline{U}_{j_0}\right|\mathrm{d}x - \frac{1}{2}\sum_{d_n(i,j)\geq 2}\int_{\Omega_R}\left|f(\overline{U}_i)\overline{U}_j\right|\mathrm{d}x \\ & + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}) \\ \geq& E_n + \beta\mathrm{e}^{-\mu g_1(R)} + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}). \end{aligned}$$

This asymptotic estimate is independent of X. □

Proposition 3.7. *The estimate*

$$\inf_{X\in\mathcal{U}_{1,R}} J_{\Omega_R}(\varphi_R(X)) \leq E_n + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3})$$

holds true as $R \to \infty$.

Proof. Fix $0 < t_1 < t_2 < \cdots < t_n < 1$ and set $x_{R,i} := R\gamma(t_i) \in \Gamma_R$. By (3.20), $X_R := (x_{R,1}, x_{R,2}, \dots, x_{R,n}) \in \mathcal{U}_{1,R}$ for large R. As in the proof of Proposition 3.6 we obtain

$$J_{\Omega_R}(\varphi_R(X_R)) = E_n - \frac{1}{2}\sum_{i\neq j}\int_{\mathbb{R}^N} f(\overline{U}_i)\overline{U}_j\,\mathrm{d}x + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}).$$

Choose $\varepsilon \in (0,\mu)$ and $\delta \in (0, \min_{i\neq j}|\gamma(t_i) - \gamma(t_j)|)$. Then $|x_{R,i} - x_{R,j}| \geq \delta R$ if $i \neq j$. Lemma 2.1 and Lemma 2.9 imply that

$$\int_{\mathbb{R}^N} f(\overline{U}_i)\overline{U}_j\,\mathrm{d}x = O(\mathrm{e}^{-(\mu-\varepsilon)\delta R}) = o(\mathrm{e}^{-\mu g_1(R)}) \qquad \text{if } i \neq j,$$

and our claim follows. □

3.2. **The open-end tube case.** We now suppose that $\gamma(0) \neq \gamma(1)$. In this case we need also to estimate the effect of the ends of the tubular domain on $V^{\pm}_{x,R}$. We start by comparing the solutions $U^{\pm}$ to the limit problem in $\mathbb{L}$ with their projections onto a finite cylinder

$$\mathbb{L}_{a,b} := (-a, b) \times B_1^{N-1}, \qquad a, b > 0.$$

Let $\widetilde{U}^{\pm}_{a,b}$ be the unique solution of

$$\begin{cases} -\Delta u + \lambda u = f(U^{\pm}) & \text{in } \mathbb{L}_{a,b}, \\ u = 0 & \text{on } \partial\mathbb{L}_{a,b}. \end{cases} \tag{3.30}$$

Again, we consider $\widetilde{U}^{\pm}_{a,b}$ to be defined in $\mathbb{R}^N$.

Lemma 3.8. *The inequalities*

$$0 \leq \widetilde{U}^{+}_{a,b}(\xi,\eta) \leq U^{+}(\xi,\eta), \qquad U^{-}(\xi,\eta) \leq \widetilde{U}^{-}_{a,b}(\xi,\eta) \leq 0, \tag{3.31}$$

and

$$|U^{\pm}(\xi,\eta) - \widetilde{U}^{\pm}_{a,b}(\xi,\eta)| \leq C_2\vartheta_{1,1}(\eta)\left(\mathrm{e}^{-\mu(a+|\xi+a|)} + \mathrm{e}^{-\mu(b+|\xi-b|)}\right) \tag{3.32}$$

hold true for all $(\xi,\eta) \in \mathbb{L}$, where C_2 is the same constant as in Lemma 2.9. *Moreover, there are $C_5, C_6 > 0$ such that*

$$C_5\mathrm{e}^{-\mu\min\{a,b\}} \leq \|U^{\pm} - \widetilde{U}^{\pm}_{a,b}\|_{H_0^1(\mathbb{L})} \leq C_6\mathrm{e}^{-\mu\min\{a,b\}}. \tag{3.33}$$

Proof. Note that $U^{\pm}$ and $\widetilde{U}^{\pm}_{a,b}$ are in $C^2(\mathbb{L}_{a,b}) \cap C(\overline{\mathbb{L}_{a,b}})$. Set $Y_{a,b} := U^{\pm} - \widetilde{U}^{\pm}_{a,b}$. We claim that the inequalities

$$\begin{aligned} C_1\vartheta_{1,1}(\eta)\max\{\mathrm{e}^{-\mu(a+|\xi+a|)}, \mathrm{e}^{-\mu(b+|\xi-b|)}\} &\leq Y_{a,b} \\ &\leq C_2\vartheta_{1,1}(\eta)(\mathrm{e}^{-\mu(a+|\xi+a|)} + \mathrm{e}^{-\mu(b+|\xi-b|)}) \end{aligned} \tag{3.34}$$

hold true for all $(\xi,\eta) \in \mathbb{L}$, where C_1 and C_2 are the constants in Lemma 2.9. This is trivially true in $\mathbb{L} \setminus \mathbb{L}_{a,b}$. For $(\xi,\eta) \in \mathbb{L}_{a,b}$ it follows from the maximum principle, because the equalities

$$\begin{aligned} (-\Delta + \lambda)\vartheta_{1,1}(\eta)\mathrm{e}^{-\mu(a+|\xi+a|)} &= 0, \\ (-\Delta + \lambda)\vartheta_{1,1}(\eta)\mathrm{e}^{-\mu(b+|\xi-b|)} &= 0, \\ (-\Delta + \lambda)Y_{a,b} &= 0, \end{aligned}$$

hold true in $\mathbb{L}_{a,b}$. Inequalities (3.31) and (3.32) are now a consequence of (3.34) and the maximum principle.

Next we prove (3.33). A straightforward computation using (3.34) yields

$$\|Y_{a,b}\|_{L^2(\mathbb{L})} = O(\mathrm{e}^{-\mu \min\{a,b\}}) \tag{3.35}$$

as $a, b \to \infty$. A standard regularity argument, cf. [12, Theorem 9.12], yields

$$\|\nabla Y_{a,b}\|_{L^2(\mathbb{L})} = O(\mathrm{e}^{-\mu \min\{a,b\}}).$$

which, together with (3.35), this gives the inequality in the right-hand side of (3.33). To prove the other inequality it is enough to show that

$$\|Y_{a,b}\|_{L^2(\mathbb{L})} \geq C\mathrm{e}^{-\mu \min\{a,b\}}, \tag{3.36}$$

where C is independent of a and b. Note that

$$a + |\xi + a| \leq b + |\xi - b| \quad \text{if and only if} \quad \xi \leq b - a.$$

It follows that

$$\begin{aligned} &\int_{\mathbb{R}} \max\{\mathrm{e}^{-\mu(a+|\xi+a|)}, \mathrm{e}^{-\mu(b+|\xi-b|)}\} \,\mathrm{d}\xi \\ &= \frac{1}{\mu}(\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu(a+b)} + \mathrm{e}^{-2\mu b}) \\ &\geq \frac{1}{\mu} \max\{\mathrm{e}^{-2\mu a}, \mathrm{e}^{-2\mu b}\} = \frac{1}{\mu}\mathrm{e}^{-2\mu \min\{a,b\}}, \end{aligned} \tag{3.37}$$

which together with (3.34) yields (3.36). The proof is complete. □

Next we compare $V^{\pm}_{x,R}$ with the function

$$W^{\pm}_{x,R}(y) := \widetilde{U}^{\pm}_{|x-R\gamma(0)|,|x-R\gamma(1)|}(A_x(y-x)), \qquad y \in \mathbb{R}^N,$$

with A_x as in (1.5) and $\widetilde{U}^{\pm}_{a,b}$ as in (3.30). Thus, the support of $W^{\pm}_{x,R}$ is contained in a copy of the finite cylinder $\mathbb{L}_{|x-R\gamma(0)|,|x-R\gamma(1)|}$, obtained by translating 0 to x and identifying $\mathbb{R} \times \{0\}$ with the tangent space to Γ_R at x.

Lemma 3.9. *For $s \in [1, r_0)$ and $p \in (0, \infty)$ the asymptotic estimates*

$$\int_{\mathbb{R}^N} |V^{\pm}_{x,s,R} - W^{\pm}_{x,R}|^p \,\mathrm{d}y = O(R^{-\min\{p,1\}}), \tag{3.38}$$

$$\int_{\mathbb{R}^N} |\nabla V^{\pm}_{x,s,R} - \nabla W^{\pm}_{x,R}|^2 \,\mathrm{d}y = O(R^{-1}), \tag{3.39}$$

$$\int_{\mathbb{R}^N} |F(V^{\pm}_{x,s,R}) - F(W^{\pm}_{x,R})| \,\mathrm{d}y = O(R^{-1}), \tag{3.40}$$

$$\int_{\mathbb{R}^N} |f(V^{\pm}_{x,s,R}) - f(W^{\pm}_{x,R})|^p \,\mathrm{d}y = O(R^{-\min\{p,1\}}), \tag{3.41}$$

hold true as $R \to \infty$, independently of $x \in \Gamma_R$.

Proof. Let $x_R \in \Gamma_R$. If x_R is far from the boundary the proof is similar to that of Lemma 3.2, but if x_R is close to the boundary the proof requires some new geometric considerations. More precisely, we consider two cases:

a) $|x_R - R\gamma(0)| \geq 2R^{1/4}$ and $|x_R - R\gamma(1)| \geq 2R^{1/4}$. Then the proof is the same as that of Lemma 3.2.

b) Either $|x_R - R\gamma(0)| < 2R^{1/4}$ or $|x_R - R\gamma(1)| < 2R^{1/4}$. Since both cases are similar, we only consider the case

$$b_R := |x_R - R\gamma(1)| < 2R^{1/4}. \tag{3.42}$$

For each R we fix a coordinate system by identifying x_R with 0 and the tangent space to Γ_R at x_R with $\mathbb{R} \times \{0\}$, preserving the orientation. In this coordinate system we consider the infinite cylinder $\mathbb{L}$ and the finite cylinders

$$\mathbb{L}_R := \mathbb{L}_{|x_R - R\gamma(0)|, |x_R - R\gamma(1)|},$$
$$Q_R := (-R^{1/3}, R^{1/3}) \times B_s^{N-1}(0),$$

and we write $R\gamma(1) = (\xi_R, \eta_R)$. Note that, since $\frac{x_R}{R} \to \gamma(1)$ as $R \to \infty$, the end of Ω_R which contains $R\gamma(0)$ lies outside of Q_R for R large enough.

We may assume that γ is defined in some interval $(0, 1+\varepsilon)$, $\varepsilon > 0$, and write $\widetilde{\Gamma}_R := \{R\gamma(t) \mid t \in [0, 1+\varepsilon)\}$ and $\widetilde{\Omega}_R$ for its tubular neighborhood of radius 1. Then $\widetilde{\Gamma}_R \cap Q_R$ is contained in the graph of a C^3-function $h_R : (-R^{1/3}, R^{1/3}) \to \mathbb{R}^{N-1}$ for large R. As before, inequalities (3.7) hold for h_R. Since $0 \le \xi_R \le b_R$ and $b_R^2 - \xi_R^2 = \eta_R^2 = h_R(\xi_R)^2$ we obtain

$$|b_R - \xi_R| = \frac{h_R(\xi_R)^2}{|b_R + \xi_R|} \le \frac{C\xi_R}{2R^2} \le C\frac{b_R}{R^2}. \tag{3.43}$$

Next, we express $\mathbb{R}^N$ as the union of the sets

$$\begin{aligned} D_R^1 &:= \mathbb{R}^N \smallsetminus Q_R, \\ D_R^2 &:= Q_R \cap \left[(\Omega_R \cup \mathbb{L}_R) \smallsetminus (\widetilde{\Omega}_R \cap \mathbb{L})\right], \\ D_R^3 &:= Q_R \cap \left[(\Omega_R \cap (\mathbb{L} \smallsetminus \mathbb{L}_R)) \cup ((\widetilde{\Omega}_R \smallsetminus \Omega_R) \cap \mathbb{L})\right], \\ D_R^4 &:= Q_R \cap \mathbb{L}_R \cap \Omega_R, \end{aligned} \tag{3.44}$$

and we show that the estimate (3.38) holds true for the integral over each one of these sets. Note that $D_R^2 \subset Q_R \cap [(\widetilde{\Omega}_R \cup \mathbb{L}) \smallsetminus (\widetilde{\Omega}_R \cap \mathbb{L})]$. Thus, the arguments for D_R^1 and D_R^2 are the same as those given to prove Claims 1-3 in Lemma 3.2. To prove estimate (3.38) over D_R^3, first observe that the angle α_R between $\{b_R\} \times \mathbb{R}^{N-1}$ and the end of Ω_R which contains $R\gamma(1)$ is the same as the angle between the tangent space to Γ_R at x_R, which we have identified with $\mathbb{R} \times \{0\}$, and the tangent space to $\widetilde{\Gamma}_R$ at $R\gamma(1)$. Therefore, using (3.7) we obtain that

$$\tan \alpha_R = |h_R'(\xi_R)| \le C\frac{b_R}{R}. \tag{3.45}$$

Since $\operatorname{diam}(B_1^{N-1}) = 2$ it follows that

$$\begin{aligned} D_R^3 &\subset [\xi_R - 2\tan\alpha_R,\, b_R + 2\tan\alpha_R] \times B_1^{N-1} \\ &\subset [b_R - s_R, b_R + s_R] \times B_1^{N-1}, \end{aligned} \tag{3.46}$$

where $s_R \ge 0$ satisfies

$$s_R \le C\left(\frac{b_R}{R^2} + \frac{b_R}{R}\right) \le C\frac{b_R}{R}. \tag{3.47}$$

Here we have used (3.43) and (3.45). Therefore, using Lemma 3.1 we conclude that

$$\begin{aligned} \int_{D_R^3} |V_{x,s,R}^{\pm} - W_{x,R}^{\pm}|^p \,\mathrm{d}y &\le C \int_{b_R - s_R}^{b_R + s_R} \mathrm{e}^{-pC_4\xi} \,\mathrm{d}\xi \\ &= C\mathrm{e}^{-pC_4 b_R} \sinh(pC_4 s_R) \\ &\le C\mathrm{e}^{-pC_4 b_R} \frac{b_R}{R} = O(R^{-1}). \end{aligned} \tag{3.48}$$

for R large enough. To prove estimate (3.38) over D_R^4, we start by estimating $|V_{x,s,R}^{\pm} - W_{x,R}^{\pm}|$ on ∂D_R^4. If $(\xi,\eta) \in \partial D_R^4 \cap \partial\mathbb{L}$ then, as in (3.12), we have that

$$\operatorname{dist}((\xi,\eta), \partial\widetilde{\Omega}_R) \le C\frac{1+\xi^2}{R}.$$

Similarly, if $(\xi,\eta) \in \partial D_R^4 \cap \partial\widetilde{\Omega}_R$ then

$$\operatorname{dist}((\xi,\eta), \partial\mathbb{L}) \le C\frac{1+\xi^2}{R}.$$

Moreover, if $(\xi,\eta) \in \partial D_R^4 \cap \partial\Omega_R \cap \widetilde{\Omega}_R$ then

$$\operatorname{dist}((\xi,\eta), \partial\mathbb{L}_R \cap \mathbb{L}) \le 2s_R \le C\frac{b_R}{R} \le C\frac{\xi + s_R}{R} \le C\left(\frac{\xi}{R} + \frac{b_R}{R^2}\right) \le C\frac{1+\xi^2}{R}.$$

Similarly, if $(\xi,\eta) \in \partial D_R^4 \cap \partial\mathbb{L}_R \cap \mathbb{L}$ then

$$\operatorname{dist}((\xi,\eta), \partial\Omega_R \cap \widetilde{\Omega}_R) \le C\frac{1+\xi^2}{R}.$$

Since $V_{x,s,R}^{\pm} = 0$ in $\mathbb{R}^N \smallsetminus \Omega_R$ and $W_{x,R}^{\pm} = 0$ in $\mathbb{R}^N \smallsetminus \mathbb{L}_R$, arguing as in the proof of Claim 4 of Lemma 3.2, we conclude that

$$|V_{x,s,R}^{\pm} - W_{x,R}^{\pm}| \le C\mathrm{e}^{-C_4|\xi|}\,\frac{1+\xi^2}{R} \qquad \text{on } \partial D_R^4, \tag{3.49}$$

and that

$$\int_{D_R^4} |V_{x,s,R}^{\pm} - W_{x,R}^{\pm}|^p \,\mathrm{d}y = O(R^{-p}). \tag{3.50}$$

This finishes the proof of (3.38).

The proof of (3.39) is analogous to that of (3.4), using the partition (3.44). Equations (3.40) and (3.41) follow from (3.38) as in the proof of Lemma 3.2. □

Again, we consider functions $g_m\colon \mathbb{R}^+ \to \mathbb{R}^+$ (to be fixed later) satisfying (3.18)-(3.20), but this time we define $D_{m,R}$ as the set of points $(x_1, x_2, \dots, x_n)$ in $(\Gamma_R)^n$ such that either there exist $i,j \in \{1,2,\dots,n\}$ with $i \ne j$ and $|x_i - x_j| \le g_m(R)$, or there exists $i \in \{1,2,\dots,n\}$ with $2\operatorname{dist}(x_i, \partial\Gamma_R) \le g_m(R)$. Then we define

$$\mathcal{U}_{m,R} := \{(x_1, x_2, \dots, x_n) \in (\Gamma_R)^n \smallsetminus D_{m,R} \mid (x_1, x_2, \dots, x_n) \text{ is an } n\text{-chain}\}. \tag{3.51}$$

Lemma 3.10. *The estimates*

$$\sup_{x\in\Gamma_R} \left\| V_{x,R}^{\pm} - W_{x,R}^{\pm} \right\|_{H_0^1(\mathbb{R}^N)} = O(R^{-1/2}), \tag{3.52}$$

$$\sup_{x\in\Gamma_R} \left| J_{\Omega_R}(V_{x,R}^{\pm}) - J_{\mathbb{L}}(W_{x,R}^{\pm}) \right| = O(R^{-1}), \tag{3.53}$$

$$\sup_{\substack{x\in\Gamma_R\\ \operatorname{dist}(x,\partial\Gamma_R)\ge g_2(R)/2}} \left\| \nabla J_{\Omega_R}(V_{x,R}^{\pm}) \right\|_{H_0^1(\Omega_R)} = O(R^{-1/2}) + O(\mathrm{e}^{-\min\{p_1,2\}\mu g_2(R)/2}) \tag{3.54}$$

hold true as $R \to \infty$.

Proof. Estimates (3.52) and (3.53) follow immediately from Lemma 3.9. To prove (3.54) we first observe that $|tU^{\pm} + (1-t)\widetilde{U}_{a,b}^{\pm}| \le |U^{\pm}|$ for every $t \in [0,1]$ by (3.31).

Moreover, (H3) implies that $|f'(u)| \le C\,|u|^{p_1-1}$ for some constant C which depends only on an upper bound for $|u|$. Therefore Lemma 2.9 and inequality (3.32) imply

$$\begin{aligned}\int_{\mathbb{L}} |f(U^\pm) - f(\widetilde{U}^\pm_{a,b})|^2 \,\mathrm{d}x &\le \int_{\mathbb{L}} \left(\int_0^1 |f'(tU^\pm) + (1-t)\widetilde{U}^\pm_{a,b}|\,\mathrm{d}t\right)^2 |U^\pm - \widetilde{U}^\pm_{a,b}|^2 \,\mathrm{d}x \\ &\le C \int_{\mathbb{R}} \mathrm{e}^{-2(p_1-1)\mu} \mathrm{e}^{-2\mu(a+|\xi+a|)} + \mathrm{e}^{-2\mu(b+|\xi-b|)} \,\mathrm{d}\xi \\ &= O(\mathrm{e}^{-2\min\{p_1,2\}\mu\min\{a,b\}}),\end{aligned}$$

as $a, b \to \infty$. Therefore,

$$\begin{aligned}\left\|f(U^\pm_{x,R}) - f(V^\pm_{x,R})\right\|_{L^2} &\le \left\|f(U^\pm_{x,R}) - f(W^\pm_{x,R})\right\|_{L^2} + \left\|f(W^\pm_{x,R}) - f(V^\pm_{x,R})\right\|_{L^2} \\ &\le O(\mathrm{e}^{-\min\{p_1,2\}\mu g_2(R)/2}) + O(R^{-1/2}),\end{aligned}$$

as $R \to \infty$. Arguing as in the proof of (3.17), using this estimate, we obtain (3.54). □

Define $\varphi_R\colon \mathcal{U}_{2,R} \to H^1_0(\Omega_R)$ by

$$\varphi_R(X) := \sum_{i=1}^{k} (V^+_{x_{2i-1},R} + V^-_{x_{2i},R}) + (n-2k)V^+_{x_n,R}, \qquad X = (x_1, x_2, \dots, x_n), \tag{3.55}$$

where k is the largest integer smaller than or equal to $\frac{n}{2}$. This time we do not require that n is even.

Next we show that the statements of Propositions 3.5–3.7 are also true for these new data. We set $\overline{U}_i$ and $\overline{V}_i$ as in (3.25). Similarly, we set

$$\overline{W}_i := \begin{cases} W^+_{x_i,R} & \text{if } i \text{ is odd}, \\ W^-_{x_i,R} & \text{if } i \text{ is even}. \end{cases}$$

Proposition 3.11. *Let α be as in* Lemma 2.2 *and fix $\alpha' \in (1/2, \min\{\alpha, p_1/2, 1\})$. Then*

$$\sup_{X\in\mathcal{U}_{2,R}} \|\nabla J_{\Omega_R}(\varphi_R(X))\|_{H^1_0(\Omega_R)} = O(\mathrm{e}^{-\alpha'\mu g_2(R)}) + O(R^{-1/2})$$

as $R \to \infty$.

Proof. The proof is completely analogous to that of Proposition 3.5, using this time Lemmas 3.10 and 3.9. □

Set

$$E_n := k\left[J_{\mathbb{L}}(U^+) + J_{\mathbb{L}}(U^-)\right] + (n-2k)J_{\mathbb{L}}(U^+).$$

Proposition 3.12. *There exists $\beta > 0$ such that*

$$\inf_{X\in\partial\mathcal{U}_{1,R}} J_{\Omega_R}(\varphi_R(X)) \ge E_n + \beta\mathrm{e}^{-\mu g_1(R)} + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3})$$

as $R \to \infty$.

Proof. Let $X = (x_1, x_2, \dots, x_n) \in \partial\mathcal{U}_{1,R}$. The proof is similar to that of Proposition 3.6 except that now we must replace $\overline{U}_i$ by $\overline{W}_i$. So, in order to arrive to the conclusion, we need the following estimates:

$$J_{x_i+A^{-1}_{x_i}\mathbb{L}}(\overline{W}_i) \ge J_{x_i+A^{-1}_{x_i}\mathbb{L}}(\overline{U}_i) + C\mathrm{e}^{-2\mu\,\mathrm{dist}(x_i,\partial\Gamma_R)}, \tag{3.56}$$

$$\int_{\mathbb{R}^N} f(\overline{W}_i)\overline{W}_j = \int_{\mathbb{R}^N} f(\overline{U}_i)\overline{U}_j + o(\mathrm{e}^{-\mu g_1(R)}). \tag{3.57}$$

Let us prove the first one. After an appropriate change of coordinates $\overline{U}_i$ becomes $U^{\pm}$ and $\overline{W}_i$ becomes $\widetilde{U}^{\pm}_{a,b}$. Recall that $J'_{\mathbb{L}}(U^{\pm}) = 0$ and observe that $|f'(U^{\pm}(\xi,\eta))| \leq C\mathrm{e}^{-(p_1-1)\mu|\xi|}$ due to condition (H3) and Lemma 2.9. So using Lemma 3.8 we obtain

$$\begin{aligned}
J_{\mathbb{L}}(\widetilde{U}^{\pm}_{a,b}) &= J_{\mathbb{L}}(U^{\pm}) + \frac{1}{2}J''_{\mathbb{L}}(U^{\pm})[\widetilde{U}^{\pm}_{a,b} - U^{\pm}, \widetilde{U}^{\pm}_{a,b} - U^{\pm}] + o(\|U^{\pm} - \widetilde{U}^{\pm}_{a,b}\|^2_{H^1_0(\mathbb{L})}) \\
&\geq J_{\mathbb{L}}(U^{\pm}) + \frac{1}{2}\|U^{\pm} - \widetilde{U}^{\pm}_{a,b}\|^2_{H^1_0(\mathbb{L})} + o(\|U^{\pm} - \widetilde{U}^{\pm}_{a,b}\|^2_{H^1_0(\mathbb{L})}) \\
&\geq J_{\mathbb{L}}(U^{\pm}) + C\mathrm{e}^{-2\mu\min\{a,b\}}
\end{aligned}$$

for R large enough. This proves (3.56).

To prove the second estimate it suffices to show that

$$\int_{\mathbb{R}^N} (f(\overline{W}_i) - f(\overline{U}_i)\overline{W}_j = o(\mathrm{e}^{-\mu g_1(R)}) \tag{3.58}$$

$$\int_{\mathbb{R}^N} f(\overline{U}_i)(\overline{W}_j - \overline{U}_j) = o(\mathrm{e}^{-\mu g_1(R)}) \tag{3.59}$$

as $R \to \infty$. Since the proof of both estimates is similar, we only prove (3.58). After a change of coordinates we may assume that $x_i = 0$ and that the tangent space to Γ_R at x_i is $\mathbb{R} \times \{0\}$. Then we set $a := |R\gamma(0)|$ and $b := |R\gamma(1)|$. We may assume without loss of generality that $a \leq b$. Since $\left|\overline{W}_j(x)\right| \leq C\mathrm{e}^{-\mu|x-x_j|}$ by (3.31) and Lemma 2.9, the proof of (3.58) reduces to showing that

$$\int_{\mathbb{L}} \left| f(U^{\pm}(x)) - f(\widetilde{U}^{\pm}_{a,b}(x))\right| \mathrm{e}^{-\mu|x-x_j|}\mathrm{d}x = o(\mathrm{e}^{-\mu g_1(R)}) \tag{3.60}$$

as $R \to \infty$. We distinguish two cases: If $|x_j| \geq 2g_1(R)$, using condition (H3), Lemma 2.1 and (3.19) we obtain

$$\begin{aligned}
\int_{\mathbb{L}} \left| f(U^{\pm}(x)) - f(\widetilde{U}^{\pm}_{a,b}(x))\right| \mathrm{e}^{-\mu|x-x_j|}\mathrm{d}x &\leq C\int_{\mathbb{L}} \mathrm{e}^{-p_1\mu|x|}\mathrm{e}^{-\mu|x-x_j|}\mathrm{d}x \\
&\leq C\mathrm{e}^{-\mu|x_j|} \leq C\mathrm{e}^{-2\mu g_1(R)} = o(\mathrm{e}^{-\mu g_1(R)})
\end{aligned}$$

as $R \to \infty$. On the other hand, if $|x_j| \leq 2g_1(R)$ we write $x_j = (\xi_j, \eta_j)$ and use the Lipschitz continuity of f on bounded sets and (3.32) to obtain

$$\begin{aligned}
&\int_{\mathbb{L}} \left| f(U^{\pm}(x)) - f(\widetilde{U}^{\pm}_{a,b}(x))\right| \mathrm{e}^{-\mu|x-x_j|}\mathrm{d}x \\
&\leq C\int_{\mathbb{R}} \left(\mathrm{e}^{-\mu(a+|\xi+a|)} + \mathrm{e}^{-\mu(b+|\xi-b|)}\right) \mathrm{e}^{-\mu|\xi-\xi_j|}\mathrm{d}\xi \\
&\leq C\left(\mathrm{e}^{-\mu(a+|\xi_j+a|)} + \mathrm{e}^{-\mu(|\xi_j-b|)}\right) \\
&= C\mathrm{e}^{-\mu(a+|\xi_j+a|)} + o(\mathrm{e}^{-\mu g_1(R)})
\end{aligned}$$

The last equality follows from $|x_j| \leq 2g_1(R)$, $b := |R\gamma(1)|$ and (3.20). Now, if $j > i$ we have that $a \geq \frac{3}{2}g_1(R)(1+o(1))$, and if $j < i$ we have that $\xi_j + a \geq \frac{3}{2}g_1(R)(1+o(1))$ as $R \to \infty$. So in both cases $\mathrm{e}^{-\mu(a+|\xi_j+a|)} = o(\mathrm{e}^{-\mu g_1(R)})$. This proves (3.60) and, hence, (3.58).

Now we may argue as in Proposition 3.6. The analogue of (3.29) with $\overline{U}_i$ replaced by $\overline{W}_i$ is obtained in a similar way. Therefore, using estimates (3.56), (3.57) and

(3.23) we conclude that

$$\begin{aligned}
J_{\Omega_R}(\varphi_R(X)) =& \sum_{i=1}^{n} J_{\Omega_R}(\overline{V}_i) + \frac{1}{2}\sum_{i\neq j}\int_{\Omega_R} f(\overline{U}_i)\overline{V}_j\,\mathrm{d}x - \sum_{i\neq j}\int_{\Omega_R} f(\overline{V}_i)\overline{V}_j\,\mathrm{d}x \\
&+ o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}) \\
=& \sum_{i=1}^{n} J_{x_i+A_{x_i}^{-1}\mathbb{L}}(\overline{W}_i) + \frac{1}{2}\sum_{i\neq j}\int_{\Omega_R} f(\overline{U}_i)\overline{W}_j\,\mathrm{d}x - \sum_{i\neq j}\int_{\Omega_R} f(\overline{W}_i)\overline{W}_j\,\mathrm{d}x \\
&+ o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}) \\
\geq& E_n + C\sum_{i=1}^{n} \mathrm{e}^{-2\mu\mathrm{dist}(x_i,\partial\Gamma_R)} + \frac{1}{2}\sum_{|i-j|=1}\int_{\Omega_R} \left|f(\overline{U}_i)\overline{U}_j\right|\,\mathrm{d}x \\
&+ o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}).
\end{aligned}$$

Since $X \in \partial\mathcal{U}_{1,R}$, either $\mathrm{dist}(x_1,\partial\Gamma_R) = g_1(R)/2$ or $\mathrm{dist}(x_n,\partial\Gamma_R) = g_1(R)/2$ or $|x_{i+1}-x_i| = g_1(R)$ for some $i = 1,...,n-1$. In any case, our claim follows. □

Proposition 3.13. *The estimate*

$$\inf_{X\in\mathcal{U}_{1,R}} J_{\Omega_R}(\varphi_R(X)) \leq E_n + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3})$$

holds true as $R \to \infty$.

Proof. The proof is similar to that of Proposition 3.7, this time taking into account that $\mathrm{dist}(x_i,\partial\Gamma_R) \geq CR$ for some $C > 0$ and every R and i. □

4. Proof of the Main Results

4.1. **The Finite Dimensional Reduction.** Let $\mathcal{U}_{2,R}$ and $\varphi_R : \mathcal{U}_{2,R} \to H_0^1(\Omega_R)$ be as in (3.21) and (3.24) when Γ is a closed curve and as in (3.51) and (3.55) if $\gamma(0) \neq \gamma(1)$. Set

$$\Sigma_R := \varphi_R(\mathcal{U}_{2,R}).$$

Lemma 4.1. *Σ_R is a finite dimensional C^2-submanifold of $H_0^1(\Omega_R)$.*

Proof. It is easy to see that the map φ_R is a C^2-immersion. If $\partial\Gamma \neq \emptyset$ or $n \leq 2$ then φ_R is injective, and hence Σ_R is a submanifold of $H_0^1(\Omega_R)$. On the other hand, if $\partial\Gamma = \emptyset$ and $n \geq 4$ then φ_R is not injective: two points in $\mathcal{U}_{2,R}$ have the same image under φ_R if and only if one of them is obtained from the other after a finite number of shifts of the form $x_i \mapsto x_{i+2}$. Since the group of permutations acts freely on $\mathcal{U}_{2,R}$, Σ_R is a submanifold of $H_0^1(\Omega_R)$ also in this case. □

We shall reduce the problem of finding a critical point of J_{Ω_R} to that of finding a critical point of a function $G_R : \Sigma_R \to \mathbb{R}$, which will be defined below.

For $u \in \Sigma_R$ we denote by $T_u\Sigma_R$ the tangent space to Σ_R at u, by $T_u^\perp\Sigma_R$ its orthogonal complement in $H_0^1(\Omega_R)$ and by $P_{u,R}^\perp : H_0^1(\Omega_R) \to T_u^\perp\Sigma_R$ the orthogonal projection. We consider $\mathrm{D}^2J_{\Omega_R}(u)$ as the derivative of the gradient vector field $\nabla J_{\Omega_R} : H_0^1(\Omega_R) \to H_0^1(\Omega_R)$ at u, and define

$$L_{u,R} := P_{u,R}^\perp \mathrm{D}^2 J_{\Omega_R}(u)|_{T_u^\perp\Sigma_R} : T_u^\perp\Sigma_R \to T_u^\perp\Sigma_R.$$

We write $\mathcal{L}(T_u^\perp\Sigma_R)$ for the space of bounded linear operators from $T_u^\perp\Sigma_R$ into itself.

Lemma 4.2. *If R is large enough and $u \in \Sigma_R$, then $L_{u,R}$ is invertible in $\mathcal{L}(T_u^\perp \Sigma_R)$ and*

$$\limsup_{R\to\infty} \sup_{u\in\Sigma_R} \left\| L_{u,R}^{-1} \right\|_{\mathcal{L}(T_u^\perp \Sigma_R)} < \infty.$$

Proof. The proof of this fact is standard, see for example Lemma 3.8(v) in [1]. □

Lemma 4.3. *There exist $r_0 > 0$ and $R_1 \geq 1$ such that for $R \geq R_1$ and for every $u \in \Sigma_R$ there is a unique $v_u \in u + T_u^\perp \Sigma_R$ which satisfies $\|u - v_u\|_{H_0^1(\Omega_R)} < r_0$ and $P_{u,R}^\perp \nabla J_{\Omega_R}(v_u) = 0$. The estimates*

$$\|u - v_u\|_{H_0^1(\Omega_R)} = O(\|\nabla J_{\Omega_R}(u)\|_{H_0^1(\Omega_R)}) \tag{4.1}$$

and

$$|J_{\Omega_R}(u) - J_{\Omega_R}(v_u)| = O(\|\nabla J_{\Omega_R}(u)\|^2_{H_0^1(\Omega_R)}) \tag{4.2}$$

hold true as $R \to \infty$, independently of $u \in \Sigma_R$. Moreover, the operator

$$P_{u,R}^\perp \mathrm{D}^2 J_{\Omega_R}(v_u)|_{T_u^\perp \Sigma_R} : T_u^\perp \Sigma_R \to T_u^\perp \Sigma_R$$

is invertible in $\mathcal{L}(T_u^\perp \Sigma_R)$.

Proof. Along this proof $B_r Z$ will denote the open ball of radius r centered at 0 in a normed space Z, and $\overline{B}_r Z$ will denote its closure.

By Lemma 4.2 we may fix $M \geq 1$ satisfying

$$M > \limsup_{R\to\infty} \sup_{u\in\Sigma_R} \left\| L_{u,R}^{-1} \right\|_{\mathcal{L}(T_u^\perp \Sigma_R)}. \tag{4.3}$$

Clearly,

$$C_0 := \limsup_{R\to\infty} \sup_{u\in\Sigma_R} \|u\|_{H_0^1(\Omega_R)} < \infty.$$

Condition (H3) yields

$$\limsup_{R\to\infty} \|J_{\Omega_R}\|_{C^{2,\bar\alpha}(B_{2C_0} H_0^1(\Omega_R))} < \infty \tag{4.4}$$

for some $\bar\alpha \in (0,1]$.

By Lemma 4.2 and (4.4) there is $r_0 > 0$ such that for R large enough

$$\left\| \mathrm{D}^2 J_{\Omega_R}(u) - \mathrm{D}^2 J_{\Omega_R}(v) \right\|_{\mathcal{L}(H_0^1(\Omega_R))} \leq \frac{1}{2M} \tag{4.5}$$

and $P_{u,R}^\perp \mathrm{D}^2 J_{\Omega_R}(v)|_{T_u^\perp \Sigma_R}$ is invertible in $\mathcal{L}(T_u^\perp \Sigma_R)$, for every $u \in \Sigma_R$ and $v \in H_0^1(\Omega_R)$ with $\|u - v\|_{H_0^1(\Omega_R)} \leq r_0$. Moreover, for R large enough,

$$\sup_{u\in\Sigma_R} \|\nabla J_{\Omega_R}(u)\|_{H_0^1(\Omega_R)} \leq \frac{r_0}{2M}. \tag{4.6}$$

because of Propositions 3.5 and 3.11. Fix $u \in \Sigma_R$ and define $g\colon T_u^\perp \Sigma_R \to T_u^\perp \Sigma_R$ by

$$g(w) := w - L_{u,R}^{-1} P_{u,R}^\perp \nabla J_{\Omega_R}(u + w).$$

If $w \in \overline{B}_{r_0} T_u^\perp \Sigma_R$ it follows from (4.5) and (4.6) that

$$\begin{aligned} \|g(w)\| &\leq M \left\| \mathrm{D}^2 J_{\Omega_R}(u) w - \nabla J_{\Omega_R}(u + w) \right\| \\ &= M \left\| -\nabla J_{\Omega_R}(u) - \int_0^1 (\mathrm{D}^2 J_{\Omega_R}(u + tw) - \mathrm{D}^2 J_{\Omega_R}(u)) w \,\mathrm{d}t \right\| \\ &\leq M \left(\|\nabla J_{\Omega_R}(u)\| + \frac{\|w\|}{2M} \right) \leq r_0. \end{aligned} \tag{4.7}$$

Hence, g maps $\overline{B}_{r_0}T_u^{\perp}\Sigma_R$ into itself. Moreover, by (4.3) and (4.5) we have

$$\|g'(w)\| \le \left\|L_{u,R}^{-1}\right\| \left\|\mathrm{D}^2 J_{\Omega_R}(u) - \mathrm{D}^2 J_{\Omega_R}(u+w)\right\| \le \frac{1}{2}.$$

Therefore, g is a contraction on $\overline{B}_{r_0}T_u^{\perp}\Sigma_R$ and by Banach's fixed point theorem g has a unique fixed point $w_u \in \overline{B}_{r_0}T_u^{\perp}\Sigma_R$. Thus, $v_u := u + w_u$ is then the only zero of $P_{u,R}^{\perp}\nabla J_{\Omega_R}$ in $u + \overline{B}_{r_0}T_u^{\perp}\Sigma_R$.

Inequality (4.7) with $w := w_u = g(w_u)$ yields $\|w_u\| \le 2M\,\|\nabla J_{\Omega_R}(u)\|$ and hence (4.1). Moreover, since $\mathrm{D}J_{\Omega_R}(v_u)[w_u] = 0$,

$$\begin{aligned}(4.8)\qquad &|J_{\Omega_R}(u) - J_{\Omega_R}(v_u)|\\ &\le |\mathrm{D}J_{\Omega_R}(v_u)[w_u]| + \int_0^t (1-t)\left|\mathrm{D}^2 J_{\Omega_R}(u+(1-t)w_u)\,[w_u,w_u]\right|\,\mathrm{d}t\\ &\le C\,\|w_u\|^2\end{aligned}$$

for some constant C independent of R and u. Now (4.1) and (4.8) imply (4.2). Finally, if R is large enough, (4.1) implies the strict inequality $\|u - v_u\|_{H_0^1(\Omega_R)} < r_0$, as stated in the lemma. □

We now fix r_0 and R_1 as in Lemma 4.3. If $R \ge R_1$ we define $G_R\colon \Sigma_R \to \mathbb{R}$ by

$$G_R(u) := J_{\Omega_R}(v_u).$$

where v_u is given by Lemma 4.3.

Proposition 4.4. *For $R \ge R_1$ the map G_R is in $C^1(\Sigma_R,\mathbb{R})$. If $u \in \Sigma_R$ is a critical point of G_R then v_u is a critical point of J_{Ω_R}.*

Proof. The map $u \mapsto v_u$ is a cross section of the normal disc bundle of radius r_0 over Σ_R, so its image $\tilde{\Sigma}_R := \{v_u : u \in \Sigma_R\}$ is a submanifold which is transversal to the fibres, that is, $H_0^1(\Omega) = T_{v_u}\tilde{\Sigma}_R \oplus T_u^{\perp}\Sigma_R$. The map $\psi_R : \Sigma_R \to \tilde{\Sigma}_R$ given by $\psi_R(u) := v_u$ is a C^1-diffeomorphism. Therefore G_R is of class C^1 and, since $\mathrm{D}G_R(u) = \mathrm{D}J_{\Omega_R}(v_u) \circ \mathrm{D}\psi_R(u)$, we have that $\mathrm{D}J_{\Omega_R}(v_u)w = 0$ for every $w \in T_{v_u}\tilde{\Sigma}_R$ if u is a critical point of G_R. But v_u was chosen so that $\mathrm{D}J_{\Omega_R}(v_u)z = 0$ for every $z \in T_u^{\perp}\Sigma_R$. Hence, v_u is a critical point of J_{Ω_R} if u is a critical point of G_R. □

4.2. **The proof of Theorems 1.1 and 1.2.** From Propositions 3.5, 3.6 and 3.7 if $\gamma(0) = \gamma(1)$, or from Propositions 3.11, 3.12 and 3.13 if $\gamma(0) \ne \gamma(1)$, and estimate (4.2), we obtain

$$\begin{aligned}\min_{X\in\partial\mathcal{U}_{1,R}} G_R(\varphi_R(X)) \ge& E_n + \beta \mathrm{e}^{-\mu g_1(R)} + o(\mathrm{e}^{-\mu g_1(R)})\\ &+ O(R^{-2/3}) + O(\mathrm{e}^{-2\alpha'\mu g_2(R)})\\ \min_{X\in\overline{\mathcal{U}_{1,R}}} G_R(\varphi_R(X)) \le& E_n + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}) + O(\mathrm{e}^{-2\alpha'\mu g_2(R)}).\end{aligned}$$

We set

$$g_1(R) := \frac{1}{2\mu}\log R \qquad \text{and} \qquad g_2(R) := \left(\frac{1}{2} + \frac{1}{4\alpha'}\right) g_1(R).$$

Since $\alpha' > 1/2$, these functions satisfy (3.18), (3.19), and (3.20). Note that

$$R^{-2/3} = o(\mathrm{e}^{-\mu g_1(R)}) \qquad \text{and} \qquad \mathrm{e}^{-2\alpha'\mu g_2(R)} = o(\mathrm{e}^{-\mu g_1(R)}).$$

Therefore,

$$\min G_R(\varphi_R(\overline{\mathcal{U}_{1,R}})) < \min G_R(\varphi_R(\partial\mathcal{U}_{1,R}))$$

if R is large enough. It follows that G_R has a local minimum $w_R := \varphi_R(X_R)$ in $\varphi_R(\mathcal{U}_{1,R}) \subset \Sigma_R$. Hence, by Lemma 4.4, $u_R := v_{w_R}$ is a critical point of J_{Ω_R}.

Moreover, by (4.1), we have that $u_R = \varphi_R(X_R) + o(1)$ in $H_0^1(\Omega_R)$ as $R \to \infty$. This, together with estimates (3.15), (3.52) and (3.33), yields (1.7) and (1.8).

Finally, (3.19) implies that $|x_{R,i} - x_{R,j}| \to \infty$ if $i \neq j$ and that $\mathrm{dist}(x_{R,i}, \partial\Gamma_R) \to \infty$ for all i, as $R \to \infty$. The proofs of Theorems 1.1 and 1.2 are complete. □

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