Superstable manifolds of semilinear parabolic problems

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Abstract

We investigate the dynamics of the semiflow φ induced on $H_0^1(\Omega)$ by the Cauchy problem of the semilinear parabolic equation

$$\partial_t u - \Delta u = f(x,u)$$

on Ω. Here $\Omega \subseteq \mathbb{R}^N$ is a bounded smooth domain, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ has subcritical growth in u and satisfies $f(x,0) \equiv 0$. In particular we are interested in the case when f is definite superlinear in u. The set

$$\mathcal{A} := \{\, u \in H_0^1(\Omega) \mid \varphi^t(u) \to 0 \text{ as } t \to \infty \,\}$$

of attraction of 0 contains a decreasing family of invariant sets

$$W_1 \supseteq W_2 \supseteq W_3 \supseteq \dots$$

distinguished by the rate of convergence $\varphi^t(u) \to 0$. We prove that the W_k's are global submanifolds of $H_0^1(\Omega)$, and we find equilibria in the boundaries $\overline{W_k} \setminus W_k$. We also obtain results on nodal and comparison properties of these equilibria. In addition the paper contains a detailed exposition of the semigroup approach for semilinear equations, improving earlier results on stable manifolds and asymptotic behavior near an equilibrium.

Contents

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1. Introduction

We are interested in parabolic Cauchy problems of the form

$$\text{(P)}\qquad \begin{cases} \partial_t u(t,x) - \Delta u(t,x) = f(x,u(t,x)) & t>0,\ x\in\Omega \\ u(t,x)=0 & t>0,\ x\in\partial\Omega \\ u(0,x)=u_0(x) & x\in\Omega \end{cases}$$

where $N \geq 1$, and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. The nonlinearity $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ has subcritical growth. Our focus is on the case that f is definite superlinear at infinity, i.e.

$$\frac{f(x,u)}{u} \to \infty \qquad \text{as } |u|\to\infty, \text{ for } x\in\Omega.$$

We consider initial data u_0 in $H_0^1(\Omega)$ and various subspaces. The precise hypotheses on f will be stated below. A model nonlinearity is

$$f(x,u) = a_0(x)u + \sum_{j=1}^{k} a_j(x)|u|^{p_j-2}u \tag{1.1}$$

with a_j in $L_\infty(\Omega)$ for $j=0,\dots,k$, $a_k(x)\geq\delta$ with some constant $\delta>0$, $2<p_1<p_2<\dots<p_k<2^*$, where $2^* := 2N/(N-2)$ if $N\geq 3$ and $2^* := \infty$ if $N=1,2$.

Our hypotheses on f imply that (P) induces a (local) semiflow φ on $H_0^1(\Omega)$. Due to the superlinear growth of f the dynamics of (P) have several interesting and challenging features. It is well known that for every $u\in H_0^1(\Omega)\setminus\{0\}$ there exists $\zeta(u)>0$ such that the solution $\varphi^t(\zeta u)$ of (P) with $u_0=\zeta u$ blows up in finite time provided $\zeta>\zeta(u)$. This blow up phenomenon has been investigated by many people; see for instance [8,39] and the references therein. As a consequence of the blow-up phenomenon there cannot exist a global attractor, the problem is not dissipative.

For the long-time dynamics the set of bounded solutions or, more generally, the set

$$\mathcal{I}_+ := \{\, u \in H_0^1(\Omega) \mid \varphi^t(u) \text{ is defined for all } t \geq 0 \,\}$$

obviously plays an important rôle. It contains the set of equilibria as well as all orbits which converge towards the set of equilibria, especially all heteroclinic orbits between equilibria. The equilibria of (P) are the time-independent solutions of the elliptic Dirichlet problem

$$\text{(E)} \qquad \begin{cases} -\Delta u(x) = f(x, u(x)) & x \in \Omega \\ \quad u(x) = 0 & x \in \partial\Omega. \end{cases}$$

There are plenty of results concerning the solution structure of (E). This is particularly true for the class of superlinear nonlinearities considered in this paper which has been a focus of research in nonlinear analysis, motivated by various applications. For this class, variational methods often yield the existence of many positive, negative, or sign-changing solutions under various hypotheses on the nonlinearity f or on the domain Ω. Standard references are the monographs [13, 41, 46, 48]. In [4] Ambrosetti and Rabinowitz showed that if f is odd as in (1.1) then (E) has an unbounded sequence of solutions. Recently it has been proved that in the odd case there even exists an unbounded sequence of nodal equilibria which are pairwise non-comparable; cf. [6].

Due to the complexity of the set of equilibria of (P) with a superlinear nonlinearity, a detailed analysis of the dynamics seems to be out of reach, at least in the higher dimensional case $N \geq 2$. Even for $N = 1$ most papers only deal with dissipative problems. In the one-dimensional case the zero number plays an important rôle for structuring the dynamics, see [9, 10, 20] for results in this direction. Unfortunately there is no generalization of the zero number to higher dimensions. Concerning the dynamics of (P) without dimensional restrictions, in addition to the papers on the blow up of solutions many authors worked on regularity problems (cf. the recent monographs [3, 31]), on the convergence of bounded solutions towards equilibria (cf. [19, 23, 24, 27, 30]), or on the structure of a global attractor or of compact isolated invariant sets as in the Chafee-Infante problem (cf. the monographs [22, 25, 42, 43, 47], and the references therein).

In the situation we are interested in, the function $u \equiv 0$ is a (trivial) equilibrium which may be unstable and degenerate. Let

$$\mathcal{A} := \{\, u \in \mathcal{I}_+ \mid \varphi^t(u) \to 0 \text{ as } t \to \infty \,\}$$

be the set of attraction of 0. If 0 is asymptotically stable then it is an open subset of $H_0^1(\Omega)$. In the case which we treat it need not even be a submanifold of $H_0^1(\Omega)$. It is our goal to present a fine analysis of the dynamics in $\overline{\mathcal{A}}$. This is quite delicate and technical in the general situation considered in this paper. In particular we investigate the set of equilibria in the boundary $\partial\mathcal{A} := \overline{\mathcal{A}} \setminus \mathcal{A}$ of $\mathcal{A}$. In a sequel we plan to study heteroclinic orbits in $\partial\mathcal{A}$. In order to give an idea of the kind of results which we obtain set

$$E := H_0^1(\Omega) \ ,$$

endowed with the norm

$$\|u\| := \left(\int_\Omega |\nabla u|^2\, dx\right)^{1/2}.$$

Let $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ be the distinct Dirichlet eigenvalues of the linearized operator $L := -\Delta - f_u(x,0)$. Let E_k^- be the generalized eigenspace of L associated to $\{\lambda_1, \dots, \lambda_{k-1}\}$ and E_k^+ the complementary eigenspace in E. For each $k \geq k_0 := \min\{j \in \mathbb{N} \mid \lambda_j > 0\}$ we consider the *k-th superstable manifold*

$$W_k := \{\, u \in \mathcal{I}_+ \mid \limsup_{t\to\infty} \|\varphi^t(u)\|^{1/t} \leq e^{-\lambda_k} \,\} \subset \mathcal{A}$$

and its boundary $D_k := \overline{W_k} \setminus W_k$. If $k_0 = 1$ then $W_1 = \mathcal{A}$ is the set of attraction of 0 for φ. We have $W_{k+1} \subset W_k$, and $D_{k+1} \subset D_k$ for all $k \geq k_0$. The sets W_k have been considered before in the case $N = 1$ for dissipative problems, see e.g. [9, 21, 49]; and see [33] where the equivalent for periodic equations is used.

The goal of the present paper is to investigate the structure of these superstable manifolds, and to find signed or nodal equilibria in D_k. Here we call a function u *signed* if either $u \geq 0$ or $u \leq 0$, and *nodal* or *sign-changing* if u is not signed. Typical results which we prove are:

- W_k is a submanifold of E with codimension $\dim E_k^-$.
- If $\lambda_2 > 0$, then W_2 is the graph of a C^1-function $U \to E_2^-$ where U is an open neighborhood of 0 in E_2^+.
- $\bigcap_{k\geq k_0} \overline{W_k} = \{0\}$
- If $k \geq 2$ then every $u \in \overline{W_k} \setminus \{0\}$ is nodal.
- If $\lambda_1 > 0$ then there exist a positive and a negative equilibrium in the boundary D_1 of W_1.
- If $\lambda_2 > 0$ then there exists a (nodal) equilibrium in D_2.
- If f is odd in u as in the model case (1.1), then for each $k \geq k_0$ there exists an equilibrium in D_k.

Using the zero number we have more results if the domain is one-dimensional. For instance, we prove that W_k is a graph for all $k \geq k_0$, and that there exists an equilibrium $u_k \in D_k$ with precisely k nodal domains, again for all $k \geq k_0$ (no oddness required).

Our approach owes a lot to the papers [10, 11, 35, 40] by Brunovský, Fiedler, Poláčik, Quittner. The usual techniques as in [25] for proving that the stable manifold of a hyperbolic equilibrium is indeed a manifold do not suffice to show that W_k is a submanifold. Observe that the third statement above implies $\bigcap_{k\geq k_0} D_k = \varnothing$, hence in the odd case there are infinitely many equilibria in the boundaries D_k. As a consequence of the fourth statement these are necessarily nodal. We shall also prove that they are unbounded and pairwise non-comparable.

Thus we have a completely new proof for the results in [4] and [6] about (E) in the odd case. In addition we obtain a great deal of information on the global dynamics of (P).

The paper is organized as follows. In the rest of this section we formulate our assumptions on f and fix notation. Then in Section 2 we investigate the structure of the superstable manifolds. Our results about equilibria on the boundary of the superstable manifolds are being stated and proved in Section 3. The proofs use the semigroup theory for semilinear parabolic problems. Standard references for these foundations are the books [3, 18, 25, 31]. Unfortunately, in the literature many of the results which we need have not been proved in sufficient generality. Other results seem to be folklore but were never written up in detail or precise hypotheses are missing. Therefore we include a rather lengthy appendix where we give a precise formulation of the semigroup setting which we use, and where we present the proofs of all results for which we did not find a reference. The results from the appendix are also needed for further investigating the dynamics in $\partial\mathcal{A}$ and D_k, especially for the existence of heteroclinic orbits between the equilibria whose existence we prove. We believe that the appendix will also be useful for other work on semilinear parabolic problems, in particular for those with superlinear nonlinearity.

1.1. The setting

In order to formulate our hypotheses on $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ we set $F(x,u) := \int_0^u f(x,s)\,ds$ and recall the critical exponent

$$2^* = \begin{cases} \frac{2N}{N-2} & N > 2 \\ \infty & N = 1, 2\ . \end{cases}$$

Let $a_i \geq 0$ for $i = 1, 2, 3, 4$, $\theta > 2$, $p \in (2, 2^*)$, and $\bar{p} \in (2, p]$ denote constants. We consider the following hypotheses:

- (F1) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function, $f(\cdot, 0) \in L_\infty(\Omega)$, and f is continuously differentiable in the second argument for a. e. x. Moreover, $|f_u(x,u)| \leq a_1(1+|u|^{p-2})$ for $x \in \Omega$, $u \in \mathbb{R}$.
- (F2) $f(x, 0) = 0$ for all $x \in \Omega$.
- (F3) $f(x,u)\,\mathrm{sign}(u) \geq a_2|u|^{\bar{p}-1} - a_3$ and $f(x,u)u \geq \theta F(x,u) - a_4$ for $x \in \Omega$, $u \in \mathbb{R}$.
- (F4) f_u is Hölder continuous at $u = 0$, uniformly in x.

Note that problems (E) and (P) are definite superlinear at infinity if (F3) holds.

Let us assume the basic assumption (F1) for the rest of this section. It follows that the energy functional $\Phi\colon E \to \mathbb{R}$ given by

$$\Phi(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(x,u)\,dx$$

is well defined and Φ is a C^2-function. The set of critical points of Φ will be denoted by

$$K := \{\, u \in E \mid \Phi'(u) = 0 \,\}\ .$$

It is well known that weak solutions of (E) are in one-to-one correspondence with critical points of Φ, and $K \subseteq C^1(\overline{\Omega})$. If (F2) is satisfied then $0 \in K$.

In Theorem B.2 we show that (P) generates a compact continuous (local) semiflow φ on E. For every $u \in E$ we denote by $T_+(u) \in (0, \infty]$ the maximal existence time of the orbit starting at u. Then $T_+ \colon E \to (0, \infty]$ is lower semicontinuous. It is known that φ possesses Φ as a strict Lyapunov function. More precisely, if $u(t) = \varphi(t, u_0)$ is an orbit, then

$$\frac{d}{dt}\Phi(u(t)) = -\|\dot{u}(t)\|^2_{L_2(\Omega)}$$

for $t \in (0, T_+(u_0))$. Here we have written $\dot{u}(t) := \frac{d}{dt}u(t)$, and this quantity exists in $L_2(\Omega)$. Moreover, the equilibria of φ are exactly the critical points of Φ.

Now suppose for the moment that (F2) holds. Recall the sets

$$\begin{aligned} \mathcal{I}_+ &= \{\, u \in E \mid T_+(u) = \infty \,\} \\ \mathcal{A} &= \{\, u \in \mathcal{I}_+ \mid \varphi^t(u) \to 0 \text{ as } t \to \infty \,\} . \end{aligned}$$

In this situation we also consider the following assumptions:

(F5) For every $C_1 \geq 0$ there is $C_2 \geq 0$ such that if $u \in \mathcal{A}$ satisfies $\|u\| \leq C_1$, then $\|\varphi^t(u)\| \leq C_2$ for all $t \geq 0$.

(F6) If $T_+(u) < \infty$ for some $u \in E$, then $\lim_{t \nearrow T_+(u)} \Phi(\varphi^t(u)) < 0$.

Quittner showed in [39] that (F5) and (F6) are consequences of (F1) and (F3), plus an additional technical condition on $p - \bar{p}$ which is vacuous if $p = \bar{p}$. We do not know whether additional conditions are needed at all, or whether (F5) and (F6) are consequences of (F1) and (F3).

Define $\mathcal{F} \in \mathcal{L}(L_2)$ by $(\mathcal{F}u)(x) := f_u(x, 0)u(x)$. As in the introduction we denote the distinct eigenvalues of $L = -\Delta - \mathcal{F}$ in L_2 with respect to Dirichlet boundary conditions by $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$. Throughout the paper we also fix

$$k_0 = \min\{\, j \in \mathbb{N} \mid \lambda_j > 0 \,\}$$

from the introduction.

1.2. General notation

We set $\mathbb{R}^+ := (0, \infty)$ and $\mathbb{R}^+_0 := [0, \infty)$. For $q \in (1, \infty]$ we denote by $L_q(\Omega)$ the Lebesgue space of real functions on Ω with norm $|\cdot|_q$. The scalar product in $L_2(\Omega)$ is written as $(\cdot\,, \cdot)$.

For a topological vector space X of real functions we denote by $\mathcal{P}X$ the cone of functions taking values in $\mathbb{R}^+_0$. The interior of $\mathcal{P}X$ will be denoted by $\mathcal{P}_0X$. Moreover we use the notation

$$\begin{aligned} u \geq v \ &:\Leftrightarrow\ u - v \in \mathcal{P}X \\ u > v \ &:\Leftrightarrow\ u - v \in \mathcal{P}X \setminus \{0\} . \end{aligned}$$

If X is a metric space, A is a point or a subset of X, and $\rho > 0$, then we set

$$\begin{aligned} U_\rho(A, X) &:= \{\, x \in X \mid \operatorname{dist}_X(x, A) < \rho \,\} \\ B_\rho(A, X) &:= \{\, x \in X \mid \operatorname{dist}_X(x, A) \leq \rho \,\} \\ S_\rho(A, X) &:= \{\, x \in X \mid \operatorname{dist}_X(x, A) = \rho \,\} \,. \end{aligned}$$

When there is no confusion possible we sometimes omit the X-dependency. If $(X, \|\cdot\|)$ is a normed vector space and $A = 0$, we often write $U_\rho X$ instead of $U_\rho(0, X)$, and so forth.

For normed vector spaces X, Y, we denote by $\mathcal{L}(X, Y)$ the space of bounded linear maps from X to Y, endowed with the operator norm. The space of closed (possibly unbounded) linear maps will be denoted by $\mathcal{C}(X, Y)$. For $A \in \mathcal{C}(X, Y)$ we denote by $\operatorname{dom}(A) \subseteq X$ the domain of A, and by $D(A)$ the domain of A endowed with the graph norm. As usual, if $X = Y$ we write $\mathcal{L}(X) := \mathcal{L}(X, X)$ and $\mathcal{C}(X) := \mathcal{C}(X, X)$.

If $U \subseteq X$ is open, $n \in \mathbb{N}_0$ and $\mu \in (0, 1)$, we write $C^n(U, Y)$ for the space of functions that have continuous derivatives up to order n, and by $C^{n+\mu}(U, Y)$ the subspace of functions in $C^n(U, Y)$ where the n-th derivative is locally Hölder continuous with exponent μ. By $C^{n-}(U, Y)$ for $n \geq 1$ we denote the subspace of functions in $C^{n-1}(U, Y)$ where the derivative of order $(n-1)$ is locally Lipschitz. We say that $u \in C^n(U, Y)$ *uniformly on bounded subsets* if all derivatives up to order n are bounded on every bounded subset of U. A similar convention applies to spaces of Hölder and Lipschitz continuous functions.

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2. Structure of Superstable Manifolds

Throughout this section we assume the hypotheses (F1) and (F2). For $k \in \mathbb{N}$ denote by E_k the eigenspace of L corresponding to the eigenvalue λ_k. For $k \geq k_0$ set

$$\sigma_k^- := \{\, \lambda_1, \ldots, \lambda_{k-1} \,\}$$

and

$$\sigma_k^+ := \sigma(L) \setminus \sigma_k^- \,.$$

Then $\sigma_k^\pm$ are spectral sets. Let $P_k^\pm$ denote the associated spectral projections and set

$$E_k^\pm := P_k^\pm E \,.$$

Then $E_k^- = \{0\}$ if $k = k_0 = 1$.

Denote for $u \in E$ by $J(u) := [0, T_+(u))$ the maximal existence interval. The domain of φ is given by

$$\mathcal{D} := \{\, (t, u) \in \mathbb{R}_0^+ \times E \mid t \in J(u) \,\} \,.$$

For $t \geq 0$ we also set

$$\mathcal{D}_t := \{\, u \in E \mid (t, u) \in \mathcal{D} \,\} \,.$$

Note that $\mathcal{D}$ is open in $\mathbb{R}_0^+ \times E$ and $\mathcal{D}_t$ is open in E. For $t \geq 0$ we write the time-t-map as $\varphi^t : \mathcal{D}_t \to E$ and we set $\varphi^{-t} := (\varphi^t)^{-1}$.

For $M \subseteq E$ we define its *positive semiorbit*, its *negative semiorbit*, and its *orbit* by

$$\begin{aligned} \mathcal{O}_+(M) &:= \bigcup_{t \geq 0} \varphi^t(\mathcal{D}_t \cap M) , \\ \mathcal{O}_-(M) &:= \bigcup_{t \geq 0} \varphi^{-t}(M) , \\ \mathcal{O}(M) &:= \mathcal{O}_+(M) \cup \mathcal{O}_-(M) , \end{aligned}$$

respectively. As a consequence of Theorem B.2d) the time-t-maps are injective, thus the notation $\varphi^{-t}(u) \in E$ for $u \in E$ with $\varphi^{-t}(\{u\}) \neq \varnothing$ is justified. We also write $\mathcal{O}(u) := \mathcal{O}(\{u\})$ for the orbit through u. We say M *is positive invariant* if $\mathcal{O}_+(M) \subseteq M$ and M *is negative invariant* if $\mathcal{O}_-(M) \subseteq M$. We say M *is locally positive* (*negative*) *invariant* if for every $u \in M$ there is an open neighborhood U of u such that $U \cap M$ is positive (negative) invariant with respect to the restriction of φ to U. We say M *is* (*locally*) *invariant* if M is (locally) positive and negative invariant.

Recall the definition of $\mathcal{A}$ given in the introduction. From Theorem B.2 we conclude that the sets $\mathcal{A}$ and $\overline{\mathcal{A}}$ are positive invariant and

$$\inf \Phi(\overline{\mathcal{A}}) \geq 0 . \tag{2.1}$$

2.1. Basic properties

We return to the concept of the k-th global superstable manifold

$$W_k = \{ u \in \mathcal{I}_+ \mid \limsup_{t \to \infty} \|\varphi(t,u)\|^{1/t} \leq e^{-\lambda_k} \} ,$$

and its boundary

$$D_k = \overline{W_k} \setminus W_k .$$

They are defined for $k \geq k_0$. If $k_0 = 1$ then 0 is asymptotically stable, and Corollary A.11 implies that $W_1 = \mathcal{A}$ is the domain of attraction of 0 in E, which is an open connected subset of E. For W_k the first part of Theorem A.14 applies, in particular W_k is an invariant set. Moreover, by Theorem B.2e) also the second part of Theorem A.14 applies to W_k, i.e. it is an injectively immersed manifold. It is also clear that $W_{k+1} \subset W_k$. We shall prove in Theorem 2.4 that $D_{k+1} \subset D_k$ for every k.

For $k \geq \max\{k_0, 2\}$ we choose some $\gamma = \gamma_k \in (\max\{\lambda_{k-1}, 0\}, \lambda_k)$ and consider the local manifold $W_{k,\text{loc}}$ in E given by Theorem A.12 for the connected component $(\lambda_{k-1}, \lambda_k)$ of $\mathbb{R} \setminus \sigma(L)$. There are open neighborhoods $U^\pm \subseteq E_k^\pm$ of 0 and $h \in C^1(U^+, U^-)$ with $h(0) = 0$, $h'(0) = 0$, such that

$$W_{k,\text{loc}} = \{ (u, h(u)) \mid u \in U^+ \} .$$

Here we identify $E = E_k^+ \oplus E_k^- = E_k^+ \times E_k^-$. If $k = k_0 = 1$ then we set $U^+ := W_1$ and $h = 0$. By Theorem A.14

$$W_k = \mathcal{O}_-(W_{k,\mathrm{loc}}) \ . \tag{2.2}$$

For $r_k > 0$ small enough such that $B_{r_k} E_k^+ \subseteq U^+$, we set

$$\begin{aligned} U_k &:= \{ (u, h(u)) \mid u \in U_{r_k} E_k^+ \} \\ B_k &:= \{ (u, h(u)) \mid u \in B_{r_k} E_k^+ \} \\ S_k &:= \{ (u, h(u)) \mid u \in S_{r_k} E_k^+ \} \ . \end{aligned}$$

We choose r_k according to the next lemma.

2.1 Lemma. *If* $r_k > 0$ *is small enough then* $\inf \Phi(S_k) > 0$.

Proof. Let $L_\pm$ denote the restriction of $L = -\Delta - \mathcal{F}$ to $P_k^\pm L_2$. Then $\sigma(L_+) = \sigma_k^+$ and $L_+^{1/2}$ is a well defined closed operator in $P_k^+ L_2$ with domain $E_k^+ = P_k^+ E$. It is known (see e.g. [3, Lemma I.1.1.2]) that then $\|\cdot\|$ and $|L_+^{1/2} \cdot |_2$ are equivalent norms on E_k^+. For $u \in E$ denote $u^\pm = P_k^\pm u$. If $u_n \to u$ in E and each u_n is a linear combination of eigenfunctions of L, then from

$$\begin{aligned} \Phi''(0)[u_n, u_n] &= (L u_n, u_n) \\ &= (L_+ u_n^+, u_n^+) + (L_- u_n^-, u_n^-) \\ &\geq |L_+^{1/2} u_n^+|_2^2 - \max\{|\lambda_1|, |\lambda_{k-1}|\} |u_n^-|_2^2 \end{aligned}$$

it follows that there are positive constants C_1, C_2, independent of u, such that

$$\Phi''(0)[u, u] \geq C_1 \|u^+\|^2 - C_2 \|u^-\|^2 \ .$$

The claim follows from $h(0) = 0$ and $h'(0) = 0$. □

2.2 Theorem. *The k-th superstable manifold W_k is a differentiable submanifold of E with codimension* $\dim E_k^-$.

Proof. We fix k and $r_k > 0$ such that the conclusion of Lemma 2.1 holds. For $\Sigma \subseteq \mathbb{R}_0^+$ we consider the set

$$M_k(\Sigma) := \bigcup_{t \in \Sigma} \varphi^{-t}(W_{k,\mathrm{loc}}) \ .$$

For one-point sets $\Sigma = \{t\}$ we write $M_k(t) := M_k(\{t\})$. Now we define

$$\widetilde{D}_k := \bigcap_{t \geq 0} \overline{W_k \setminus M_k([0,t])} \ .$$

If $u \in W_k$, by Theorem A.12c) and (2.2) there is $t \geq 0$ such that $\varphi(t, u) \in B_k$. We can therefore define $\tau \colon W_k \to \mathbb{R}_0^+$ by

$$\tau(u) := \min\{ t \geq 0 \mid \varphi(t, u) \in B_k \} \ . \tag{2.3}$$

Since $W_{k,\mathrm{loc}}$ is locally negative invariant by Theorem A.12b), $\varphi(\tau(u), u) \in S_k$ for $u \in W_k \setminus B_k$. It follows from Lemma 2.1 that

$$\inf \Phi(\overline{W_k \setminus B_k}) > 0 \; . \tag{2.4}$$

We claim that

$$\widetilde{D}_k \cap W_k = \varnothing \; . \tag{2.5}$$

To show this, suppose we are given $u \in \widetilde{D}_k \cap W_k$. Then there are sequences $(u_n) \subseteq W_k$ and $(t_n) \subseteq \mathbb{R}_0^+$ with $u_n \to u$ and $t_n \to \infty$, such that $\varphi(t_n, u_n) \in W_k \setminus W_{k,\mathrm{loc}}$. From (2.4) it follows that

$$\delta := \inf_n \Phi(\varphi(t_n, u_n)) > 0 \; .$$

On the other hand there is $t_0 \geq 0$ such that $\Phi(\varphi(t_0, u)) < \delta$. Hence $\Phi(\varphi(t_0, u_n)) < \delta$ for large n, contradicting the definition of δ since $t_n \to \infty$. This proves (2.5).

From (2.2) it is clear that

$$W_k = \bigcup_{t \geq 0} M_k([0, t]) \; . \tag{2.6}$$

Set $m := \dim E_k^-$. The arguments in the proof of Theorem A.14 show that $M_k([0, t])$ is an m-codimensional submanifold of E for all $t \geq 0$. Now suppose that $u \in W_k$. By (2.5) there are $r > 0$ and $t \geq 0$ such that

$$U_r(u) \cap W_k = U_r(u) \cap M_k([0, t]) \; .$$

Since $u \in W_k$ was arbitrary, W_k is an m-codimensional differentiable submanifold of E. □

The next theorem contains several properties of superstable manifolds which are important for our approach to the existence of equilibria in the boundaries of the superstable manifolds. They are also of some independent interest. Let $\mathcal{I}$ denote the set of $u \in E$ such that $\varphi^t(u)$ exists for all $t \in \mathbb{R}$ and $\mathcal{O}(u)$ is bounded. Note that $\alpha(u) \neq \varnothing \neq \omega(u)$ for $u \in \mathcal{I}$ due to the compactness of the semiflow. Here $\alpha(u)$ and $\omega(u)$ denote the α- and ω-limit sets of u, respectively. Now we define

$$K_1 := \{ u \in K \setminus \{0\} \mid \exists v \in \mathcal{I} \colon u \in \alpha(v),\ \omega(v) = \{0\} \} \; .$$

The set K_1 consists of those nontrivial equilibria of φ that possess a (generalized) connecting orbit to 0.

Before we state the theorem, we note a simple consequence of Theorems B.2 and A.3:

2.3 Lemma. *Suppose that* (F5) *holds. Then* $\overline{\mathcal{A}} \subseteq \mathcal{I}_+$. *Moreover, if* $M \subseteq \overline{\mathcal{A}}$ *is precompact, then* $\mathcal{O}_+(M)$ *is precompact.*

2.4 Theorem. *Consider* $k \geq k_0$. *Then the following hold:*

a) *D_k is closed in E, positive invariant with respect to φ, and* $\inf \Phi(D_k) > 0$. *If $k_0 \le k \le k'$, then $D_{k'} \subseteq D_k$.*

b) *If $(t_n) \subseteq \mathbb{R}_0^+$ and $(u_n) \subseteq W_k$ satisfy $t_n \to \infty$, $u_n \to u$ for some $u \in E_k$, and $\varphi(t_n, u_n) \in S_k$ for all n, then $u \in D_k$.*

c) *Assume* (F5). *If $D_k \ne \varnothing$ then $K_1 \cap D_k \ne \emptyset$. More precisely, given $u \in D_k$ and a sequence $(u_n) \subseteq W_k$ with $u_n \to u$ as $n \to \infty$ we have:*

$$K_1 \cap D_k \cap \overline{\mathcal{O}_+(\{u_n\}_n)} \ne \varnothing \ .$$

d) *If* (F4) *holds, then* $\bigcap_{k \ge k_0} W_k = \{0\}$. *If* (F4) *and* (F5) *hold, then* $\bigcap_{k \ge k_0} D_k = \varnothing$.

Proof. We use the same notation as in the proof of Theorem 2.2. First we show that

$$D_k \subseteq \widetilde{D}_k = \bigcap_{t \ge 0} \overline{W_k \setminus M_k([0,t])} \ . \tag{2.7}$$

Pick $u \in D_k$ and a sequence $(u_n) \subseteq W_k$ such that $u_n \to u$. It suffices to show that for every $t \ge 0$ there is n_0 such that $u_n \notin M_k([0,t])$ for $n \ge n_0$. If this is not the case, we may assume that $(u_n) \subseteq M_k([0,t])$ for some $t \ge 0$. For some $t_0 \ge 0$ by Theorem A.12c) we have $\varphi(t_0, M_k([0,t])) \subseteq B_k$. Therefore $\tau(u_n) \le t_0$ for all n where $\tau(u)$ is as in (2.3). Hence we may also assume that $\tau(u_n) \to t_1$ as $n \to \infty$. Then $\varphi(\tau(u_n), u_n) \to \varphi(t_1, u) \in B_k$. This contradicts the choice of u and thus (2.7) is proved. Implicitly we have also proved

$$(u \in D_k,\ (u_n) \subseteq W_k,\ u_n \to u \text{ in } E) \quad \Longrightarrow \quad \tau(u_n) \to \infty \ . \tag{2.8}$$

a) From $\widetilde{D}_k \subseteq \overline{W_k}$, (2.5) and (2.7) we conclude that

$$D_k = \widetilde{D}_k \tag{2.9}$$

and that D_k is closed in E. Moreover, $D_k \subseteq \overline{W_k \setminus W_{k,\text{loc}}} \subseteq \overline{W_k \setminus B_k}$ and (2.4) yield

$$\inf \Phi(D_k) > 0 \ . \tag{2.10}$$

It follows from the continuity of φ that D_k is positive invariant. Let us consider $k_0 \le k \le k'$. In view of (2.10), and by positive invariance, $D_{k'} \cap W_k = \varnothing$. Hence $D_{k'} \subseteq D_k$.

b) In this situation $u \in \overline{W_k}$. Assume that $u \in W_k$. Then $\varphi(t_0, u) \in U_k$ for some $t_0 \ge 0$ and thus we may assume that $\varphi(t_0, u_n) \in U_k$ for all n. By Theorem A.12c) there is t_1 such that $\varphi([t_1, \infty), u_n) \subseteq U_k$ for all n, contradicting the properties of (u_n). Hence $u \in D_k$ and b) is shown.

c) Suppose that we are given $u \in D_k$ and $(u_n) \subseteq W_k$ with $u_n \to u$. We may assume that $u_n \in W_k \setminus B_k$ so that $v_n := \varphi(\tau(u_n), u_n) \in S_k$. By Lemma 2.3 $\mathcal{O}_+(\{u_n\}_n)$ is precompact so $v_n \to v \in S_k$, possibly after passing to a subsequence. We fix $t \ge 0$ and observe that (2.8) implies $\tau(u_n) \ge t$ for n large. By compactness we may assume that $\varphi(\tau(u_n) - t, u_n)$ converges to some

$$v_t \in \overline{\mathcal{O}_+(\{u_n\}_n)} \ .$$

Now

$$\varphi(t, v_t) = \lim_{n\to\infty} \varphi(t, \varphi(\tau(u_n) - t, u_n)) = \lim_{n\to\infty} \varphi(\tau(u_n), u_n) = v \ .$$

Moreover

$$\varphi(t, v) = \lim_{n\to\infty} \varphi(t, \varphi(\tau(u_n), u_n)) = \lim_{n\to\infty} \varphi(\tau(u_n) + t, u_n) \in \overline{\mathcal{O}_+(\{u_n\}_n)} \ .$$

Since $t \geq 0$ was arbitrary, these observations prove that $v \in \mathcal{I}$ and

$$\mathcal{O}(v) \subseteq \overline{\mathcal{O}_+(\{u_n\}_n)} \ .$$

Hence $\alpha(v) \neq \emptyset$, and from $\omega(v) = \{0\}$ it follows that

$$\alpha(v) \subseteq D_k \cap \overline{\mathcal{O}_+(\{u_n\}_n)} \ .$$

This proves c).

d) If

$$u \in \bigcap_{k\geq k_0} W_k \setminus \{0\}$$

then $\lim_{t\to\infty} \|\varphi(t, u)\|^{1/t} = 0$, in contradiction with Lemma B.4b). Therefore

$$\bigcap_{k\geq k_0} W_k = \{0\} \ . \tag{2.11}$$

Now suppose that

$$u \in \bigcap_{k\geq k_0} D_k \ .$$

There are $u_k \in W_k$ such that $u_k \to u$ as $k \to \infty$. The proof of c) yields an element $v \in S_{k_0} \cap \mathcal{I}$ with

$$v \in \bigcap_{k\geq k_0} \overline{\mathcal{O}_+(\{u_n\}_{n\geq k})} \subseteq \bigcap_{k\geq k_0} \overline{W_k} \ .$$

From a) it follows that

$$v \in \bigcap_{k\geq k_0} W_k \ ,$$

contradicting (2.11). We conclude that

$$\bigcap_{k\geq k_0} D_k = \emptyset$$

which together with (2.11) finishes the proof of d). □

2.5 Remark. From a technical viewpoint it is also interesting to consider the semiflow in the space $H^1_{q,0}(\Omega)$, the closure of the set of C^∞-functions with compact support in Ω in the

Sobolev space $H^1_q(\Omega)$ of order 1 and exponent $q \geq 2$. This is done in Section B to prove regularity results. One can also define

$$W_{q,k} := \{ u \in \mathcal{I}_+ \cap H^1_{q,0}(\Omega) \mid \limsup_{t\to\infty} \|\varphi(t,u)\|^{1/t}_{H^1_{q,0}} \leq e^{-\lambda_k} \}$$

and $D_{q,k} := \overline{W_{q,k}} \setminus W_{q,k}$ for $q \geq 2$ and $k \geq k_0$. It follows from Lemma B.4a) that then

$$W_{q,k} = W_{2,k} \cap H^1_{q,0}(\Omega) \ . \tag{2.12}$$

Since by Theorem 2.2 $W_{2,k} = W_k$ is a submanifold of E of finite codimension, (2.12) and the denseness of the embedding $H^1_{q,0}(\Omega) \hookrightarrow E$ imply that $W_{q,k}$ is a submanifold of $H^1_{q,0}(\Omega)$ of the same finite codimension. If $2 \leq q' \leq q''$ then (2.12) and the continuity of the embedding $H^1_{q'',0}(\Omega) \hookrightarrow H^1_{q',0}(\Omega)$ imply

$$D_{q'',k} \subseteq D_{q',k} \cap H^1_{q'',0}(\Omega) \ . \tag{2.13}$$

Moreover, $D_{q,k}$ is closed in $H^1_{q,0}(\Omega)$. To see this, assume the contrary. Then there are $u \in W_{q,k}$ and $(u_n) \subseteq D_{q,k}$ such that $u_n \to u$ in $H^1_{q,0}(\Omega)$ and hence also in E. By (2.13) with $q' = 2$ and $q'' = q$, and by the closedness of $D_{2,k}$ in E given in Theorem 2.4a), $u \notin W_{2,k}$. This contradicts (2.12), and hence $D_{q,k}$ must be closed.

2.2. Nodal properties and comparison results

In this subsection in addition to (F1) and (F2) we assume (F4).

2.6 Theorem. *No two distinct elements of $\overline{W_k}$ are comparable if $k \geq 2$, that is, $u_1 - u_2$ changes sign for $u_1, u_2 \in \overline{W_k}$, $u_1 \neq u_2$. In particular, every $u \in \overline{W_k} \setminus \{0\}$ changes sign.*

Proof. Assume first that there are $u_1, u_2 \in W_k$ with $v_0 := u_1 - u_2 > 0$. By the comparison principle Theorem B.2c) $v(t) := \varphi(t,u_1) - \varphi(t,u_2) > 0$ for all $t \geq 0$. Hence by Lemma B.4b) we can apply Corollary A.11 and obtain that $v(t)/\|v(t)\|$ approaches the compact set

$$M := S_1 E_{k_1}$$

for some $k_1 \geq k$. Since E_{k_1} is orthogonal to E_1 in L_2, and $E_1 \subseteq \mathcal{P}E \cup (-\mathcal{P}E)$, every function in E_{k_1} changes sign. Moreover, $\mathcal{P}E$ is closed, so that $\operatorname{dist}(M, \mathcal{P}E \cup (-\mathcal{P}E)) > 0$, contradicting $v(t)/\|v(t)\| \in \mathcal{P}E$. Therefore $v_0 > 0$ is not possible. For the general case, assume that $u_1, u_2 \in \overline{W_k}$ and $v_0 := u_1 - u_2 > 0$. There are sequences $(u_{i,n})_n \subseteq W_k$ $(i = 1, 2)$ converging to u_i as $n \to \infty$. Applying φ^t to this setting, with some small $t > 0$, by the comparison principle we may assume that v_0 lies in $\mathcal{P}_0 C^1(\overline{\Omega})$, and by invariance and Theorem B.2a) that $u_{i,n} \to u_i$ in $C^1(\overline{\Omega})$. Hence $u_{1,n} - u_{2,n} > 0$ for large n, which we have shown to be impossible. □

As a corollary we obtain that W_2 is a graph.

2.7 Theorem. *If $\lambda_2 > 0$ then the restriction of P_2^+ to W_2 is a diffeomorphism onto an open neighborhood U of 0 in E_2^+. Stated differently, W_2 is the graph of a C^1-function $U \to E_2^-$.*

Proof. Theorem 2.6 implies that the restriction $P_2^+|_{W_2}$ is injective. Since $\ker P_2^+ = E_2^- = E_1 \subseteq \mathcal{P}E \cup (-\mathcal{P}E)$ it remains to show that every nontrivial tangent vector of W_2 is a sign changing function. Assume that we are given $u \in W_2$ and $v_0 \in T_u W_2 \cap \mathcal{P}E \setminus \{0\}$. Set $v(t) := D\varphi^t(u)v_0$. By Theorem A.14

$$\limsup_{t\to\infty} \|v(t)\|^{1/t} \leq e^{-\lambda_2} .$$

Repeating the arguments above we see that this cannot happen, i.e. every tangent vector is a nodal function as claimed. □

2.8 Remark. In the special case $\lambda_1 > 0$, using the order structure, Poláčik [35] defined $W_2 \cap H^1_{q,0}(\Omega)$ and implicitly proved Theorem 2.7. Here $H^1_{q,0}(\Omega)$ denotes the closure of C^∞-functions with compact support in Ω in the Sobolev space $H^1_q(\Omega)$ of order 1 with exponent $q > N$.

We can obtain some information about the location of certain weak super- and subsolutions of (E) relative to $\overline{\mathcal{A}}$. Denote by $C^2_0(\overline{\Omega})$ the space of functions in $C^2(\overline{\Omega})$ that vanish on $\partial\Omega$. Define

$$\mathcal{S}^+_{\text{reg}} := \{\, u \in C^2_0(\overline{\Omega}) \mid -\Delta u(x) \geq f(x, u(x)) \text{ for } x \in \Omega \,\}$$
$$\mathcal{S}^-_{\text{reg}} := \{\, u \in C^2_0(\overline{\Omega}) \mid -\Delta u(x) \leq f(x, u(x)) \text{ for } x \in \Omega \,\}$$

and

$$\mathcal{S}^\pm := \overline{\mathcal{S}^\pm_{\text{reg}}}$$

where the closure is taken in E. Hence $\mathcal{S}^+_{\text{reg}}$ ($\mathcal{S}^-_{\text{reg}}$) is the set of regular supersolutions (subsolutions) for problem (E). The set $\mathcal{S}^+$ ($\mathcal{S}^-$) consists entirely of weak supersolutions (subsolutions) of (E), respectively. We do not know if the sets $\mathcal{S}^\pm$ are *exactly* the weak super- and subsolutions of (E).

2.9 Theorem. *Suppose that $u_1 \in \overline{\mathcal{A}}$. If $u_2 \in \mathcal{S}^+$ and $u_2 > u_1$, then $u_2 \geq 0$. Similarly, if $u_2 \in \mathcal{S}^-$ and $u_2 < u_1$, then $u_2 \leq 0$.*

Proof. We restrict our attention to the case that $u_2 \in \mathcal{S}^-$ and $u_2 < u_1$. If $u_1 \in \mathcal{A}$ then $0 \in u_2 + \mathcal{P}E$ by Lemma B.3, which proves the claim in this particular situation. In the general case, fix $t \in (0, T_+(u_2))$ and let $(v_n) \subseteq \mathcal{A}$ be a sequence that converges to u_1 in E. Then $w_n := \varphi^t(v_n) \in \mathcal{A}$ converges to $\varphi^t(u_1)$ in $C^1(\overline{\Omega})$ and moreover, $\varphi^t(u_1) - \varphi^t(u_2) \in \mathcal{P}_0 C^1(\overline{\Omega})$. Hence, for n large enough, we have $w_n > \varphi^t(u_2) \geq u_2$, again by Lemma B.3. Since we have already handled this situation above, the proof is complete. □

The last three theorems can be considerably improved if $N = 1$, so $\Omega \subset \mathbb{R}$ is an open bounded interval. An important tool is the zero number which we recall here. For a continuous function $h\colon \overline{\Omega} \to \mathbb{R}$ not vanishing everywhere, define the *zero number* $z(h) \in \mathbb{N}_0 \cup \{\infty\}$ of h to be the supremum of all $n \in \mathbb{N}_0$ such that there is a strictly increasing sequence $x_0 < x_1 < x_2 < \ldots < x_n$ in $\overline{\Omega}$ with

$$h(x_{i-1})h(x_i) < 0 \qquad \text{for } i = 1, \ldots, n\ .$$

Since in one space dimension $E \subseteq C(\overline{\Omega}, \mathbb{R})$ we have $z\colon E \setminus \{0\} \to \mathbb{N}_0 \cup \{\infty\}$. For further properties of the zero number we refer the reader to [5, 9] and the references therein.

2.10 Theorem. *Suppose* $N = 1$. *Then* $z(u_1 - u_2) \geq k - 1$ *for two distinct elements* $u_1, u_2 \in \overline{W_k}$. *In particular,* $z(u) \geq k - 1$ *for every* $u \in \overline{W_k} \setminus \{0\}$.

Proof. For $k \in \mathbb{N}$ it is known that the zero number z satisfies

$$\text{(2.14)} \qquad \begin{cases} z(u) \leq k - 2 & \text{if } k \geq 2,\ u \in E_k^- \setminus \{0\} \\ z(u) \geq k - 1 & \text{if } u \in E_k^+ \setminus \{0\}\ ; \end{cases}$$

cf. for example [37]. Moreover $z\colon E \setminus \{0\} \to \mathbb{N}_0 \cup \{\infty\}$ is lower semicontinuous. Consider $u_1, u_2 \in W_k$ with $v_0 := u_1 - u_2 \neq 0$. Defining $v(t) := \varphi(t, u_1) - \varphi(t, u_2)$ as in the proof of Theorem 2.6, it is known that $z(v(t))$ decreases in t. Hence using (2.14), similar arguments as before show that $z(v_0) \geq k - 1$. For the general case, we consider $u_1, u_2 \in \overline{W_k}$ with $v_0 := u_1 - u_2 \neq 0$, and set $T := \min\{\, T_+(u_1), T_+(u_2) \,\}$ and $v(t) := \varphi(t, u_1) - \varphi(t, u_2)$ for $t \in [0, T)$. For every $t_0 \in (0, T)$ such that there exists $x_0 \in \Omega$ with $v(t)(x_0) = 0$ and $\partial_x v(t)(x_0) = 0$ we have $z(v(t_1)) > z(v(t_2))$, for $0 \leq t_1 < t_0 < t_2 < T$; cf. [5]. Since $z(v(t)) \in \mathbb{N}_0$, there is $t_0 \in (0, T)$ such that for every $x \in \overline{\Omega}$ with $v(t_0)(x) = 0$ we have $\partial_x v(t_0)(x) \neq 0$. Thus there exists a neighborhood U of $v(t_0)$ in $C^1(\overline{\Omega})$ so that z is constant on U. A similar approximation argument as in the proof of Theorem 2.6 now shows that $z(v_0) \geq z(v(t_0)) \geq k - 1$, proving the claim. ☐

2.11 Theorem. *Suppose that* $N = 1$. *Then the restriction of* P_k^+ *to* W_k *is a diffeomorphism onto an open neighborhood* U *of* 0 *in* E_k^+. *Stated differently,* W_k *is the graph of a* C^1*-function* $U \to E_k^-$.

Proof. As in the proof of Theorem 2.7 one can show that every nontrivial tangent vector v of W_k satisfies $z(v) \geq k - 1$. Hence the discussion above together with (2.14) gives Theorem 2.11. ☐

2.12 Theorem. *Suppose that* $N = 1$, $u_1 \in \overline{\mathcal{A}}$, u_2 *is a nontrivial solution of* (E), *and* $u_1 \neq u_2$. *Then* $z(u_2) \leq z(u_2 - u_1)$.

Proof. First consider the case $u_1 \in \mathcal{A}$. For $t \geq 0$

$$z(u_2 - u_1) \geq z(\varphi(t, u_2) - \varphi(t, u_1)) = z(u_2 - \varphi(t, u_1))\ .$$

Moreover $\varphi(t, u_1) \to 0$ in $C^1(\overline{\Omega})$ as $t \to \infty$. The arguments in the proof of Theorem 2.10 imply that z is continuous in u_2 with respect to the $C^1(\overline{\Omega})$-topology. Hence $z(u_2 - u_1) \geq z(u_2)$. The general case of $u_1 \in \overline{\mathcal{A}}$ follows by approximation as in the proof of Theorem 2.10. ☐

3. Equilibria on the boundary of superstable manifolds

In this section we assume the hypotheses (F1)–(F5). Recall the set K_1 of equilibria that lie in the α-limit set of an orbit in the domain of attraction of 0. We denote by K_1^+ the intersection of K_1 with the positive cone in E, and by K_1^- the intersection with the negative cone. Thus $K_1^+ \cup K_1^-$ consists of the signed equilibria in the boundary of the domain of attraction. The set of nodal equilibria will be denoted by $K_1^* := K_1 \setminus (K_1^+ \cup K_1^-)$. By the strong maximum principle, a signed equilibrium is either strictly positive or strictly negative in Ω.

The theorems in this section will be proved in Section 3.1. We begin with the existence of signed equilibria in K_1.

3.1 Theorem. *If $\lambda_1 > 0$, then $K_1^+ \neq \varnothing$ and $K_1^- \neq \varnothing$.*

The existence of signed solutions of E is a consequence of the famous mountain pass theorem of Ambrosetti and Rabinowitz [4]. Generically we expect the solutions in Theorem 3.1 to be of mountain pass type in the sense of Hofer [28].

Surprisingly, the existence of nodal solutions of E on a general domain without any symmetry is a recent result. We refer the reader to [6, 7, 12, 17]. The first results on the existence of signed and nodal solutions in the boundary of the domain of attraction are due to Quittner [37, 38, 40] who treated the case $\lambda_1 > 0$ where 0 is asymptotically stable.

3.2 Theorem. a) *If $\lambda_2 > 0$, then $K_1^* \cap D_2 \neq \varnothing$.*

b) *Assume* (F6). *If $\lambda_2 \leq 0$ and $F(x,u) \geq (\lambda_{k_0-1} + f_u(x,0))u^2/2$ for $x \in \overline{\Omega}$ and $u \in \mathbb{R}$, then $K_1^* \cap D_{k_0} \neq \varnothing$.*

3.3 Remark. If (F3) holds with $a_4 = 0$ and if $\lambda_{k_0-1} < 0$, then it is easy to see that the additional condition in b) of the theorem above is satisfied.

The condition on F in Theorem 3.2b) implies that the energy satisfies $\Phi(u) \leq 0$ for $u \in E_{k_0}^-$. Using variational methods one can show that there exists a nodal equilibrium if $\Phi(u) \leq 0$ for $u \in E_{k_0}^-$ near 0. This local linking condition is satisfied if $\lambda_{k_0-1} < 0$, for example. However, in that case we do not know whether there is a nodal equilibrium in D_{k_0}.

It is well known that there are infinitely many equilibria when f is odd in u (cf. [4]). The existence of infinitely many nodal equilibria has been proved in [6] using variational methods. We can now find these equilibria on the boundary of the superstable manifolds.

3.4 Theorem. *Assume* (F6). *If f is odd in u, then Φ is unbounded on $K_1^* \cap D_k$ for every $k \geq k_0$. Stated differently, there exists a sequence of equilibria $\pm u_k \in K_1^* \cap D_k$ with $\Phi(u_k) \to \infty$ as $k \to \infty$.*

More can be said in the case of space dimension $N = 1$.

3.5 Theorem. *If $N = 1$, $\Omega = (0, l)$, there is a doubly infinite sequence $(u_k) \subseteq K_1$ ($k \in \mathbb{Z}$, $|k| \geq k_0$) such that $u_k \in D_{|k|} \setminus D_{|k|+1}$, $z(u_k) = |k| - 1$, $\operatorname{sign} \partial_x u_k(0) = \operatorname{sign} k$ and $\Phi(u_k) \to \infty$ as $|k| \to \infty$.*

3.6 Remark. In view of $K_1 \subseteq \partial\mathcal{A}$, Theorems 2.9 and 2.12 are applicable to the solutions of (E) we have constructed. This yields the following extremal property: Suppose that $u_1 \in K_1$. If $u_2 \in \mathcal{S}^+$ and $u_2 > u_1$, then $u_2 \geq 0$. Similarly, if $u_2 \in \mathcal{S}^-$ and $u_2 < u_1$, then $u_2 \leq 0$. This extremal property has also been proved for the solutions constructed in [6]. As a consequence we note that two distinct $u_1, u_2 \in K_1$ that are comparable must be signed with opposite sign.

In the case of $N = 1$, in addition we can say the following: If $u_1 \in K_1$, and if u_2 is a nontrivial solution of (E) different from u_1, then $z(u_2) \leq z(u_2 - u_1)$.

3.7 Remark. a) In the one-dimensional case the existence of infinitely many solutions has been proved by Struwe in [45] for more general boundary value problems. In addition to the new dynamical information contained in Theorem 3.5 also the fact that one has nodal solutions with precisely k nodal domains for each $k \geq k_0$ is not contained in Struwe's paper.

b) There are many results on nodal solutions in the radial setting, when $N \geq 2$. We refer to the paper by Conti et al. [16] for references in this direction. There one also finds a dynamic point of view based on the heat semiflow which is related to our approach. The use of zero number techniques can yield more detailed information in this case than in the nonradial case.

c) It follows from results in [32, 50] that in Theorem 3.5 the set K_1 can be replaced by $K_2 := \{u \in K \setminus \{0\} \mid \exists v \in \mathcal{I}\colon \alpha(v) = \{u\},\ \omega(v) = \{0\}\}$; see also the remarks following the statement of [20, Prop. 1.1]. It seems to be an open problem whether Theorem 3.1 holds with K_2 instead of K_1. It was shown in [11] that generically in f all equilibria are hyperbolic, hence isolated. If all equilibria are isolated then we have of course $K_1 = K_2$.

3.1. Proofs of the results about existence of equilibria

Condition (F3) ensures that Φ satisfies the Palais-Smale condition, i.e. every sequence $(u_n) \subseteq E$ such that $\Phi(u_n)$ is bounded above and $\Phi'(u_n) \to 0$ in E' is precompact. Here E' denotes the dual of E. As simple and well known consequence of (F3) and (2.1) we note without proof:

3.8 Lemma. *If Y is a finite dimensional subspace of E, then*

$$\lim_{\substack{\|u\|\to\infty \\ u\in Y}} \Phi(u) = -\infty\ .$$

Moreover $Y \cap \overline{\mathcal{A}}$ is bounded.

Proof of Theorem 3.1. Since W_1 is an open neighborhood of 0 in E, by Lemma 3.8 the set $U := W_1 \cap E_1$ is a bounded open neighborhood of 0 in E_1. It follows from the comparison principle that U is connected. Let $u^+, u^- \in D_1$ denote the boundary points of U, such that $\pm u^\pm \in \mathcal{P}E$. Pick $u_0^\pm \in \omega(u^\pm)$. This is possible by Lemma 2.3. Then $\pm u_0^\pm \in D_1 \cap K \cap \mathcal{P}E$. Now an argument as in the proof of Theorem 2.6 shows that there is $(u_n) \subseteq W_1 \cap \mathcal{P}E$

converging to u_0^+. Moreover, $\mathcal{O}_+(\{u_n\}_n) \subseteq \mathcal{P}E$ by the maximum principle, and $\mathcal{P}E$ is closed. Using this information and Theorem 2.4c) we find $K_1^+ \neq \emptyset$. The proof of $K_1^- \neq \emptyset$ is similar. □

Proof of Theorem 3.2a). By Theorems 2.4a), c) and 2.6 it suffices to show that $D_2 \neq \emptyset$. From Theorem 2.7 it follows that the restriction of P_2^+ to $W_2 \cap E_3^-$ is a diffeomorphism onto an open neighborhood U of 0 in E_2. If $D_2 = \emptyset$ then W_2 is closed, and by Lemma 3.8 $W_2 \cap E_3^-$ is compact. This contradicts the fact that U is a nonempty open subset of a finite dimensional space, finishing the proof. □

Proof of Theorem 3.2b). As in the proof of Theorem 3.2a) it suffices to show that $D_{k_0} \neq \emptyset$.

Let $h\colon U^+ \to E_{k_0}^-$ be the C^1-map defined on a neighborhood U^+ of 0 in $E_{k_0}^+$ whose graph is $W_{k_0,\mathrm{loc}}$. Put $r_- := \sup_{u \in B_{r_+}E_{k_0}^+} \|h(u)\|$ where $r_+ := r_{k_0}$ is as in Lemma 2.1. Extend $h|_{B_{r_+}E_{k_0}^+}$ by a continuous map $\tilde{h}\colon E_{k_0}^+ \to B_{r_-}E_{k_0}^-$. Pick some $w \in E_{k_0}^+$ with $\|w\| = 1$ and set $Y := E_{k_0}^- \oplus [w]$. Here $[w]$ denotes the linear hull of the set $\{w\}$. Using Lemma 3.8 choose $R > r_+ + r_-$ large enough such that $\Phi(u) \le 0$ for $u \in Y \setminus U_R Y$. In view of the assumption made we find for $u \in E_{k_0}^-$

$$
\begin{aligned}
\Phi(u) &= \frac{1}{2}(\nabla u, \nabla u) - \int_\Omega F(x, u(x))\, dx \\
&\le \frac{1}{2}(-\Delta u, u) - \frac{1}{2}\int_\Omega (\lambda_{k_0-1} + f_u(x,0))u(x)^2\, dx \\
&= \frac{1}{2}(Lu, u) - \frac{1}{2}\lambda_{k_0-1}|u|_2^2 \\
&\le 0
\end{aligned} \tag{3.1}
$$

Define

$$M := \{\, v + sw \mid v \in E_{k_0}^-,\ \|v + sw\| \le R,\ s \ge 0 \,\}$$

and denote by M_0 the boundary of M in Y. By (3.1) and the choice of R we have $\Phi(u) \le 0$ for $u \in M_0$. We claim that if $\psi\colon M \to E$ is continuous and $\psi|_{M_0} = \mathrm{id}|_{M_0}$, then $\psi(M) \cap S_{k_0} \neq \emptyset$. To see this, consider the continuous map $\kappa\colon M \to Y$ given by

$$\kappa(u) := P_{k_0}^-\psi(u) - \tilde{h}(P_{k_0}^+\psi(u)) + \tilde{h}(\|P_{k_0}^+\psi(u)\|w) + \|P_{k_0}^+\psi(u)\|w\ .$$

Clearly $\kappa|_{M_0} = \mathrm{id}|_{M_0}$ so that by a degree argument $M \subseteq \kappa(M)$. Moreover $\|r_+w + \tilde{h}(r_+w)\| \le r_+ + r_- < R$ giving $r_+w + \tilde{h}(r_+w) \in M$. Hence there is $u \in M$ with $\kappa(u) = r_+w + \tilde{h}(r_+w)$. It follows that $\|P_{k_0}^+\psi(u)\| = r_+$ and $P_{k_0}^-\psi(u) = \tilde{h}(P_{k_0}^+\psi(u))$, thus $\psi(u) \in S_{k_0}$. The claim is proved.

We need to construct a modification of the semiflow φ as follows: Define

$$\tau(u) := \inf\{\, t \in J(u) \mid \Phi(\varphi(t,u)) \le 0 \,\}\ .$$

By (F6) $T_+(u) = \infty$ if $\tau(u) = \infty$. Therefore we can set for $t \ge 0$

$$\tilde{\varphi}(t,u) := \begin{cases} \varphi(t,u) & t < \tau(u) \\ \varphi(\tau(u),u) & t \ge \tau(u)\ . \end{cases}$$

Then $\widetilde{\varphi}$ is a global continuous semiflow on E. If $\Phi(\widetilde{\varphi}(t,u)) > 0$ then $t < T_+(u)$ and $\widetilde{\varphi}(t,u) = \varphi(t,u)$. If $\Phi(u) \le 0$ then $\widetilde{\varphi}(t,u) = u$ for all $t \ge 0$. Moreover, Φ is a Lyapunov function for $\widetilde{\varphi}$.

Now suppose that $(t_n) \subseteq \mathbb{R}_0^+$ with $t_n \to \infty$ as $n \to \infty$. The construction of M and $\widetilde{\varphi}$, together with what we have proved above, shows that for each n there is $u_n \in \widetilde{\varphi}^{-t_n}(S_{k_0}) \cap M$. It holds that $\Phi(\widetilde{\varphi}(t_n, u_n)) > 0$ by Lemma 2.1. Thus $\varphi(t_n, u_n) \in S_{k_0}$ for all n. Since M is compact we may assume that $u_n \to u \in M$. Using Theorem 2.4b) we conclude that $u \in D_{k_0}$ and finish the proof. □

Proof of Theorem 3.4. First we show that

$$K_1 \cap D_k \neq \varnothing \quad \text{for } k \ge k_0 \ . \tag{3.2}$$

Therefore, fix $k \ge k_0$ and let Y be a finite dimensional subspace of E with

$$\dim Y > \dim E_k^- \ . \tag{3.3}$$

Let $h\colon U^+ \to E_k^-$ be the C^1-map defined on a symmetric neighborhood U^+ of 0 in E_k^+ whose graph is $W_{k,\mathrm{loc}}$. Put $r_- := \sup_{u \in B_{r_k} E_k^+} \|h(u)\|$ where r_k is as in Lemma 2.1. Since f is odd in u, h is an odd map also, and φ is odd in its second argument. Extend $h|_{B_{r_k} E_k^+}$ by an odd continuous map $\tilde{h}\colon E_k^+ \to B_{r_-} E_k^-$. Fix $R > r_k + r_-$ such that $\Phi(u) \le 0$ for $u \in Y \setminus U_R Y$. We claim that if $\psi\colon B_R Y \to E$ is odd and continuous, with $\psi|_{S_R Y} = \mathrm{id}|_{S_R Y}$, then $\psi(B_R Y) \cap S_k \neq \varnothing$. For a proof of this fact set

$$V := \{ u \in U_R Y \mid \|P_k^+ \psi(u)\| < r_k \} = \psi^{-1}(E_k^- + U_{r_k} E_k^+) \cap U_R Y \ .$$

Then V is a symmetric, bounded and open neighborhood of 0 in Y. Set $\Sigma := \partial_Y V$, the boundary of V in Y. It is easy to see that

$$\psi(\Sigma \cap U_R Y) \subseteq E_k^- + S_{r_k} E_k^+ \ . \tag{3.4}$$

Define $\kappa\colon \Sigma \to E_k^-$ by

$$\kappa(u) := P_k^- \psi(u) - \tilde{h}(P_k^+ \psi(u)) \ .$$

Then κ is odd and continuous. By virtue of (3.3) and the theorem of Borsuk-Ulam there is $u \in \Sigma$ with $\kappa(u) = 0$. If $u \in S_R Y$ then $\psi(u) = u$ and thus $P_k^- u = \tilde{h}(P_k^+ u)$ from $\kappa(u) = 0$. Moreover $\|P_k^- u\| \le r_-$ by the choice of $\tilde{h}$, and $\|P_k^+ u\| \le r_k$ by the definition of Σ. Hence $\|u\| \le r_k + r_- < R$, a contradiction. Therefore $u \in U_R Y$. By (3.4) $\|P_k^+ \psi(u)\| = r_k$, so that together with $\kappa(u) = 0$ we find $\psi(u) \in S_k$. The claim is proved.

As in the proof of Theorem 3.2b) it follows from (F6) and from what we have shown above that $D_k \neq \varnothing$. This proves (3.2) in view of Theorem 2.4c).

Now if Φ was bounded on $K_1 \cap D_{k_1}$ for some $k_1 \ge k_0$, then $\overline{K_1 \cap D_{k_1}}$ would be compact as a consequence of the Palais-Smale condition. Since every D_k is closed and since $D_{k+1} \subset D_k$, from (3.2) it would follow that $\bigcap_{k \ge k_0} D_k \neq \varnothing$, contradicting Theorem 2.4d). The proof is complete. □

Proof of Theorem 3.5. Recall the relations (2.14). Also recall that $\dim E_k = 1$ for all $k \in \mathbb{N}$. The existence of $u_{\pm 1}$ in the case $k = k_0 = 1$ is covered by Theorem 3.1. Fix some $k \geq \max\{ k_0, 2 \}$ and denote by e_k an eigenfunction of L for the eigenvalue λ_k such that $\partial_x e_k(0) > 0$ and $\|e_k\| = 1$.

From Theorem 2.11 we deduce the existence of an open neighborhood U of 0 in E_k^+ and a C^1-map $h\colon U \to E_k^-$, such that W_k is the graph of h. For some $r > 0$ small enough, by Theorem A.12c) the image V_{loc} of $U_r E_k^+$ under the map $u \mapsto (u, h(u))$ satisfies $\mathcal{O}_+(V_{\mathrm{loc}}) \subseteq B_k$. Since W_{k+1} is the graph of a C^1-function mapping an open neighborhood of 0 in E_{k+1}^+ into E_{k+1}^- and since $D_{k+1} \cap W_k = \varnothing$, the set $V_{\mathrm{loc}} \setminus W_{k+1}$ has exactly two connected components, open in W_k. The same holds true for $U_k \setminus W_{k+1}$. Denote the component of V_{loc} that contains $(\varepsilon e_k, h(\varepsilon e_k))$ for small $\varepsilon > 0$ by V_{loc}^+ and the other one by V_{loc}^-. If $u \in V_{\mathrm{loc}}^+$, then $\varphi(t, u)$ stays in one component of $U_k \setminus W_{k+1}$ and therefore never enters V_{loc}^-. The same holds the other way around. Now we define $V^\pm := \mathcal{O}_-(V_{\mathrm{loc}}^\pm)$. By the observation above $\{ V^+, V^- \}$ are invariant nonempty disjoint (relatively) open subsets of W_k covering $W_k \setminus W_{k+1}$.

By Corollary A.11, if $u \in W_k \setminus W_{k+1}$ then

$$\tau(u) := \lim_{t \to \infty} \frac{\varphi(t, u)}{\|\varphi(t, u)\|} \in S_1 E_k = \{ e_k, -e_k \} .$$

Clearly $\tau(u) = \pm e_k$ for $u \in V^\pm$.

The set $M = E_{k+1}^- \cap W_k$ is a 1-dimensional submanifold of E containing 0, and $M \cap \overline{W_{k+1}} = \{0\}$ by Theorem 2.10. Thus

$$z(u) = k - 1 \tag{3.5}$$

for $u \in M \setminus \{0\}$. From Lemma 3.8 we deduce that M is bounded, and it is a graph over E_k in E_{k+1}^-. We find boundary points $v^\pm \in D_k$ of the connected component of M containing 0 such that $v^\pm \in \overline{M \cap V^\pm}$.

In order to construct u_k we now restrict our attention to v^+. The construction of u_{-k} from v^- proceeds analogously. There is a sequence $(v_n) \subseteq M \cap V^+$ converging to v^+. By (3.5) and Theorem 2.4c) there exists $u_k \in K_1 \cap D_k \cap \overline{V^+}$ with $z(u_k) = k - 1$.

Next we show that $\partial_x u_k(0) > 0$. Since u_k solves (E) we have $\partial_x u_k(0) \neq 0$. Consider a sequence $(w_n) \subseteq V^+$ converging to u_k in C^1. We may therefore assume that $\partial_x w_n(0) \neq 0$ and $z(w_n) = k - 1$ for all n. For fixed n, the integer valued function $t \mapsto z(\varphi(t, w_n))$ is constant since $w_n \in W_k$. This implies that $\partial_x \varphi(t, w_n)(0)$ cannot change sign as $t \to \infty$.

Recall that since $N = 1$, with the notation of Section A.2 we may choose the densely injected Banach couple $(X_0, X_1) = (L_2(\Omega), H^2(\Omega) \cap H_0^1(\Omega))$ to analyze the semiflow. Here $L := -\Delta \in \mathcal{H}(X_1, X_0)$ and $E = X_{1/2}$. Take some $\beta \in (3/4, 1)$, then $X_\beta \hookrightarrow C^1(\overline{\Omega})$. Applying Corollary A.11c) we know that

$$\frac{\varphi(t, w_n)}{\|\varphi(t, w_n)\|_{X_\beta}} \to \frac{\tau(w_n)}{\|\tau(w_n)\|_{X_\beta}} = \frac{e_k}{\|e_k\|_{X_\beta}} \qquad \text{in } X_\beta \text{ as } t \to \infty.$$

Therefore $\partial_x \varphi(t, w_n)(0) > 0$ for large t and fixed n, and by the considerations above $\partial_x w_n(0) > 0$. This shows that $\partial_x u_k(0) > 0$.

Finally we prove that $\Phi(u_k) \to \infty$ as $|k| \to \infty$. Clearly z is continuous on $K \setminus \{0\}$ in the $C^1(\overline{\Omega})$-topology. Observe also that $(u_k) \subseteq D_{k_0}$, so that by 2.4a) (u_k) is bounded away from 0. By Theorem B.2a) the topologies of $C^1(\overline{\Omega})$ and E coincide on K. If $\Phi(u_k)$ is bounded for a subsequence of (u_k), there is a subsequence converging in K as a consequence of the Palais-Smale condition. The observations made above imply boundedness of the zero number along this subsequence, contradicting $z(u_k) \to \infty$ as $|k| \to \infty$. □

A. Abstract semilinear parabolic problems

In this appendix we prove various properties of the semiflow of abstract autonomous semilinear parabolic problems. Some results are variations of essentially known results. These are included for the convenience of the reader. In other cases we prove strengthened versions, and we give proofs for folklore statements we did not find a reference for. We refer the reader to [3, 18, 25] for general background. We also construct the superstable manifolds and prove some basic properties which hold in a more general context.

We say that (X_0, X_1) is a densely injected Banach couple if X_0 and X_1 are Banach spaces and X_1 is densely injected in X_0. By $\mathcal{H}(X_1, X_0)$ we denote the set of those $A \in \mathcal{L}(X_1, X_0)$ that are negative generators of a strongly continuous analytic semigroup on X_0 if considered as operators in X_0 with domain X_1.

Let $[\cdot,\cdot]_\alpha$ for $\alpha \in (0,1)$ denote the complex interpolation functor of exponent α. By X_α we denote either

- the fractional power space generated by the fractional powers of $A + \omega$, where $A \in \mathcal{H}(X_1, X_0)$ satisfies $\sigma(A) \subseteq [\operatorname{Re} z > -\omega]$ for some $\omega \in \mathbb{R}$

or

- the interpolation space $[X_0, X_1]_\alpha$.

The results presented here hold with either definition. For our application to the concrete problem (P) there will be no difference, since there A has bounded imaginary powers.

For convenience, if $\alpha, \beta \in [0,1]$, we use the notation $\|\cdot\|_\alpha := \|\cdot\|_{X_\alpha}$ and $\|\cdot\|_{\alpha,\beta} := \|\cdot\|_{\mathcal{L}(X_\alpha, X_\beta)}$.

A.1. Linear integral operators

Let (X_0, X_1) be a densely injected Banach couple. Fix $A \in \mathcal{H}(X_1, X_0)$, $\alpha \in [0,1)$. We write $U(t,s) := e^{-(t-s)A}$ for $s, t \in \mathbb{R}$, $s \le t$.

Since existence theory of semilinear equations is based on the variation-of-constants formula, it is convenient to state some properties of corresponding integral operators. The next lemma is a variant of [18, Lem. 5.5]. We emphasize the existence of bounds that are independent of the length of the considered interval.

A.1 Lemma. *Fix $J := [t_0, t_1]$ with $t_0 < t_1$ and define an operator H by setting for $g \in L_\infty(J, X_0)$:*

$$H(g)(t) := \int_{t_0}^{t} U(t,s) g(s)\, ds$$

if $t \in J$.

a) *If* $0 \le \alpha \le \beta < 1$ *then*

$$H \in \mathcal{L}(L_\infty(J, X_0), C^{\beta-\alpha}(J, X_\alpha)) \ .$$

b) *If* $0 < \gamma \le 1$ *then*

$$H \in \mathcal{L}(C^\gamma(J, X_0), C(J, X_1)) \ .$$

In either case, if $\sigma(A) \subseteq [\operatorname{Re} z > 0]$*, the norm of* H *is bounded independently of the length of* J.

Proof. We prove this lemma assuming that $\sigma(A) \subseteq [\operatorname{Re} z > 0]$ for simplicity. Without this assumption the statements remain true, but the constants depend on the length of J.

Choose $\omega > 0$ such that $\sigma(A) \subseteq [\operatorname{Re} z > \omega]$. Set $|J| := t_1 - t_0$. For $x < 1$, $y \ge 0$ define

$$\kappa(x, y) := \int_0^y s^{-x} e^{-\omega s} \, ds \ .$$

Then κ is monotone increasing in y. Since $\omega > 0$, for all $x \in [0, 1)$ and $y \ge 0$

$$\kappa(x, y) \le \lim_{r \to \infty} \kappa(x, r) = \Gamma(1 - x)\omega^{x-1} < \infty \ . \tag{A.1}$$

a) Put $\gamma := \beta - \alpha$, so that $0 \le \gamma < 1 - \alpha$. From [18, Prop. 6.8] it follows that

$$\|U(t, s)\|_{0,\alpha} \le C(\alpha) e^{-(t-s)\omega} (t - s)^{-\alpha}$$

for $s \le t$, so that for $t \in J$:

$$\begin{aligned} \left\| \int_{t_0}^t U(t, s) g(s) \, ds \right\|_\alpha &\le C(\alpha) \|g\|_\infty \int_{t_0}^t (t - s)^{-\alpha} e^{-(t-s)\omega} \, ds \\ &= C(\alpha) \|g\|_\infty \kappa(\alpha, t - t_0) \\ &\le C(\alpha) \|g\|_\infty \end{aligned} \tag{A.2}$$

by (A.1).

It is sufficient to consider the case $\alpha < \beta$, so that $\gamma > 0$. We follow the proof of [18, Lem. 5.5]. First we remark that for $s \in [t_0, t_1]$

$$U(\cdot, s) \in C^\gamma([s, t_1], \mathcal{L}(X_\beta, X_\alpha)) \tag{A.3}$$

with Hölder norm bounded by a constant independent of s, t_0 and t_1. This can be seen by carefully inspecting the proof of [18, Lem. 5.3(b)] and using Theorem 6.6 and Proposition 6.8 *loc. cit.*

Let $t_0 \le r < t \le t_1$. Using Proposition 6.8 *loc. cit.* and (A.3) we find for $s \in [t_0, r)$:

$$\begin{aligned} \|U(t, s) - U(r, s)\|_{0,\alpha} &\le \|U(t, r) - U(r, r)\|_{\beta,\alpha} \|U(r, s)\|_{0,\beta} \\ &\le C(\alpha, \beta)(t - r)^\gamma (r - s)^{-\beta} e^{-(r-s)\omega} \ , \end{aligned}$$

giving

$$\int_{t_0}^{r} \|U(t,s) - U(r,s)\|_{0,\alpha}\, ds \le C(\alpha,\beta)(t-r)^{\gamma} \int_{t_0}^{r} (r-s)^{-\beta} e^{-(r-s)\omega}\, ds$$
$$= C(\alpha,\beta)(t-r)^{\gamma}\, \kappa(\beta, r-t_0)$$
$$\le C(\alpha,\beta)(t-r)^{\gamma}$$

as above. Moreover

$$\int_{r}^{t} \|U(t,s)\|_{0,\alpha}\, ds \le C(\alpha) \int_{r}^{t} (t-s)^{-\alpha}\, ds = C(\alpha)(t-r)^{1-\alpha} .$$

It follows that

$$\|H(g)(t) - H(g)(r)\|_{\alpha} = \left\| \int_{t_0}^{r} (U(t,s) - U(r,s)) g(s)\, ds + \int_{r}^{t} U(t,s) g(s)\, ds \right\|_{\alpha}$$
$$\le C(\alpha,\beta)\|g\|_{\infty} \Big((t-r)^{\gamma} + (t-r)^{1-\alpha} \Big) .$$

For $|t-r| \le 1$ we thus have

$$\|H(g)(t) - H(g)(r)\|_{\alpha} \le 2C(\alpha,\beta)\|g\|_{\infty} |t-r|^{\gamma}$$

and for $|t-r| \ge 1$ we have, using (A.2),

$$\|H(g)(t) - H(g)(r)\|_{\alpha} \le 2\|H(g)\|_{\infty} \le 2\|H(g)\|_{\infty}|t-r|^{\gamma} \le 2C(\alpha,\beta)\|g\|_{\infty}|t-r|^{\gamma}$$

which proves the claim.

b) We have

$$\text{(A.4)} \qquad H(g)(t) = \int_{t_0}^{t} U(t,s)(g(s) - g(t))\, ds + (1 - U(t,t_0))A^{-1} g(t)$$

since

$$\frac{d}{ds} U(t,s) A^{-1} g(t) = U(t,s) g(t) .$$

But $A^{-1}g \colon J \to X_1$ is continuous and $U(\cdot, t_0) \colon J \to \mathcal{L}(X_1)$ is continuous with respect to the strong operator topology (see [18, Def. 2.3]). Therefore the second term on the right hand side of (A.4) is continuous as a map from J into X_1.

Let $T := t_1 - t_0$ and set $\Delta_T := \{(t,s) \mid 0 \le s \le t \le T\}$ and $\dot{\Delta}_T := \{(t,s) \mid 0 \le s < t \le T\}$. Put $a(t,s) := U(t,s)(g(s) - g(t))$ for $t, s \in J$ with $s \le t$, and $b(t,s) := a(t+t_0, s+t_0)$ for $(t,s) \in \Delta_T$. Then $b \in C(\dot{\Delta}_T, X_1)$ and

$$\|b(t,s)\|_1 \le C\|g\|_{C^{\gamma}(J,X_0)} (t-s)^{\gamma-1}$$

for $(t,s) \in \dot{\Delta}_T$ by [18, Prop. 6.8]. Define

$$v(t) := \int_0^t b(t,s)\, ds$$

for $t \in [0, T]$. Then [18, Lem. 5.8] gives us $v \in C([0, T], X_1)$, and thus

$$\int_{t_0}^{t} a(t, s)\, ds = v(t - t_0)$$

is continuous from J into X_1. We have shown that $H(g) \in C(J, X_1)$.

Lastly, from (A.4) and (A.1) we find, using Theorem 6.6 and Proposition 6.8 in [18],

$$\begin{aligned}
\|H(g)(t)\|_1 \le & \int_{t_0}^{t} \|U(t, s)\|_{0,1} \|g(s) - g(t)\|_0\, ds \\
& \qquad + \Big(1 + \|U(t, t_0)\|_{1,1}\Big) \|A^{-1}\|_{0,1} \|g(t)\|_0 \\
& \le C\|g\|_{C^\gamma(J,X_0)} \left(\int_{t_0}^{t} (t - s)^{\gamma-1} e^{-(t-s)\omega}\, ds + 1 \right) \\
& \le C\|g\|_{C^\gamma(J,X_0)} (\kappa(1 - \gamma, t) + 1) \\
& \le C(\gamma)\|g\|_{C^\gamma(J,X_0)} \, .
\end{aligned}$$

This finishes the proof. □

A.2 Corollary. *Fix $J := [t_0, t_1]$ with $t_0 < t_1$ and define an operator K by setting for $x \in X_0$ and $g \in L_\infty(J, X_0)$:*

$$K(x, g)(t) := U(t, t_0)x + \int_{t_0}^{t} U(t, s)g(s)\, ds$$

if $t \in J$.

a) If $0 \le \alpha \le \beta < 1$, then

$$K \in \mathcal{L}(X_\beta \times L_\infty(J, X_0), C^{\beta-\alpha}(J, X_\alpha)) \, .$$

b) If $0 < \gamma \le 1$ then

$$K \in \mathcal{L}(X_1 \times C^\gamma(J, X_0), C(J, X_1)) \, .$$

In either case, if $\sigma(A) \subseteq [\mathrm{Re}\, z > 0]$, the norm of K is bounded independently of the length of J.

Proof. This follows easily from Lemma A.1 using Corollary 5.4 and Theorem 6.6 in [18] together with (A.3). □

A.2. The parabolic semiflow

Suppose that (X_0, X_1) is a densely injected Banach couple. Fix $A \in \mathcal{H}(X_1, X_0)$, $\alpha \in [0, 1)$ and suppose moreover that $f : X_\alpha \to X_0$ is Lipschitz continuous, uniformly on bounded subsets.

Consider the Cauchy problem

$$\text{(A.5)} \qquad \begin{cases} \dot{u}(t) + Au(t) = f(u(t)) & t > 0 \\ u(0) = u_0 & u_0 \in X_\alpha \ . \end{cases}$$

A *solution* of (A.5) is a function $u \in C(J, X_\alpha) \cap C^1(\dot{J}, X_0)$ where $J := [0, T)$, $\dot{J} := (0, T)$, for some $T > 0$, such that $u(t) \in X_1$ for all $t \in \dot{J}$ and such that $u(0) = u_0$. A solution of (A.5) always satisfies the variation-of-constants formula

$$\text{(A.6)} \qquad u(t) = U(t, t_0)u(t_0) + \int_{t_0}^{t} U(t, s) f(u(s))\, ds \qquad t_0 \in J,\ t \in [t_0, T).$$

A *mild solution* of (A.5) is a function $u \in C(J, X_\alpha)$ that satisfies (A.6).

We also need to consider a linearized form of this equation. If $f \in C^1(X_\alpha, X_0)$, $J = [0, T)$, and $u \in C(J, X_\alpha)$, consider the Cauchy problem

$$\text{(A.7)} \qquad \begin{cases} \dot{v}(t) + Av(t) = f'(u(t))v(t) & t > 0 \\ v(0) = v_0 & v_0 \in X_\alpha \ . \end{cases}$$

It is easy to see, using the results from Section A.1, that for every $v_0 \in X_\alpha$ there is a unique mild solution $v(t)$ of (A.7), i.e. $v \in C(J, X_\alpha)$ satisfies

$$v(t) = U(t, t_0)v(t_0) + \int_{t_0}^{t} U(t, s) f'(u(s))v(s)\, ds \qquad t, t_0 \in J,\ t_0 \le t.$$

Moreover, $v \in C(\dot{J}, X_\beta)$ for every β in $[0, 1)$.

A.3 Theorem. *For every $u_0 \in X_\alpha$ there is a maximal $T_+(u_0) \in (0, \infty]$ such that setting $J := J(u_0) := [0, T_+(u_0))$ and $\dot{J} := J \setminus \{0\}$ the Cauchy problem* (A.5) *has a solution $u \in C(J, X_\alpha) \cap C^1(\dot{J}, X_0) \cap C(\dot{J}, X_1)$. These solutions induce a local continuous semiflow φ on X_α. We set*

$$\mathcal{D} := \{\, (t, u) \in \mathbb{R}_0^+ \times X_\alpha \mid t \in J(u) \,\}$$

and $\dot{\mathcal{D}} := \mathcal{D} \setminus (\{0\} \times X_\alpha)$. For $s \ge 0$ we also set

$$\mathcal{D}_s := \{\, u \in X_\alpha \mid (s, u) \in \mathcal{D} \,\} \ .$$

Then we have the following additional properties:

a) *If $\|\varphi(t, u)\|_\alpha$ is uniformly bounded on $J(u)$ for some $u \in X_\alpha$, then $T_+(u) = \infty$.*

b) *$\mathcal{D}$ is open in $[0, \infty) \times X_\alpha$ and $\mathcal{D}_s$ is open in X_α for all $s \ge 0$.*

c) *The map $T_+ \colon X_\alpha \to (0, \infty]$ is lower semicontinuous.*

d) *$\varphi \colon \mathcal{D} \to X_\alpha$ is continuous, and locally Lipschitz continuous in its second argument.*

e) *For every* $\beta \in (\alpha, 1)$, $\varphi\colon \dot{\mathcal{D}} \to X_\beta$ *is continuous, and locally Lipschitz continuous in its second argument.*

f) *If* $f \in C^1(X_\alpha, X_0)$ *uniformly on bounded subsets, then* $\varphi\colon \mathcal{D} \to X_\alpha$ *and* $\varphi\colon \dot{\mathcal{D}} \to X_\beta$ *are continuously differentiable in the second argument, for all* $\beta \in (\alpha, 1)$. *If* $u \in C(J, X_\alpha)$ *is a solution of* (A.5) *and* $v_0 \in X_\alpha$, *then* $v(t) := D\varphi^t(u(0))v_0$ *is the mild solution of* (A.7).

g) *For fixed* $T \in (0, \infty]$ *and* $V \subseteq T_+^{-1}((T, \infty) \cup \{\infty\})$, *and every* $\varepsilon \in [0, T)$ *we put*

$$M(\varepsilon) := \bigcup_{t \in [\varepsilon, T)} \varphi(t, V) \ .$$

Then, if $M(\varepsilon_1)$ *is bounded in* X_α *for some* $\varepsilon_1 \in (0, T)$, *also* $M(\varepsilon_2)$ *is bounded in* X_1 *for all* $\varepsilon_2 \in (\varepsilon_1, T)$.

A.4 Remark. The local Lipschitz property d) is to be understood as follows: For every $(t_0, x_0) \in \mathcal{D}$ there is a neighborhood U of (t_0, x_0) in $\mathcal{D}$ and a constant C such that

$$\|\varphi(t, x) - \varphi(t, y)\|_\alpha \le C\|x - y\|_\alpha$$

for all $(t, x), (t, y) \in U$. A similar remark applies to e).

As a simple consequence of the preceding theorem we note:

A.5 Corollary. *Assume that* A *has compact resolvent. If* T *and* V *are as in Theorem* A.3g) *and* $M(\varepsilon_1)$ *is bounded in* X_α *for some* $\varepsilon_1 \in (0, T)$, *then* $M(\varepsilon_2)$ *is precompact in* X_β *for all* $\beta \in [0, 1)$ *and* $\varepsilon_2 \in (\varepsilon_1, T)$. *In this case we say that* φ *is a* compact *semiflow.*

If V *is precompact in* X_α *and* $M(\varepsilon)$ *is bounded in* X_α *for all* $\varepsilon \in (0, T)$, *then* $M(0)$ *is precompact in* X_α

Before giving a sketch of the proof of Theorem A.3, we prove a technical result, a specialized and strengthened version of [18, Lem. 16.7].

A.6 Lemma. *For* $\beta \in [\alpha, 1)$ *and* $T, \rho > 0$ *there is a constant* $C = C(\alpha, \beta, \rho, T)$ *such that if* $t \in (0, T]$ *and* $u, v \in C([0, t], X_\alpha)$ *are mild solutions of* (A.5) *satisfying*

$$\sup_{s \in [0,t]} \|u(s)\|_\alpha, \ \sup_{s \in [0,t]} \|v(s)\|_\alpha \le \rho \ ,$$

then

$$\|u(t) - v(t)\|_\beta \le Ct^{\alpha-\beta}\|u(0) - v(0)\|_\alpha \ .$$

Proof. Let t, u, v be given and put $w := u - v$. For every $\tau \in [0, t]$ we estimate

$$\begin{aligned} \|w(\tau)\|_\beta &\le \|U(\tau, 0)\|_{\alpha,\beta}\|w(0)\|_\alpha + \int_0^\tau \|U(\tau, s)\|_{0,\beta}\|f(u(s)) - f(v(s))\|_0 \, ds \\ &\le C(\alpha, \beta)t^{\alpha-\beta}\|w(0)\|_\alpha + C(\beta, \rho)\int_0^\tau (\tau - s)^{-\beta}\|w(s)\|_\alpha \, ds \ . \end{aligned} \tag{A.8}$$

Setting $\beta = \alpha$ in this inequality and applying Gronwall's lemma in the form of [18, Cor. 16.6] (note that it's conclusion also holds on subintervals with a uniform constant) we find

$$\|w(s)\|_\alpha \leq C(\alpha, \beta, \rho, T)\|w(0)\|_\alpha$$

for $s \in [0, t]$. Plugging this inequality into (A.8) with $\tau = t$ proves the lemma. □

Proof of Theorem A.3. The existence of a unique solution of (A.5) and of the associated semi-flow with properties a)–c) and f) is proved in [18, Sects. 15,16]. d) is proved in a slightly weaker form in [18, Thm. 16.8], but the proof is easily extended to yield our statement d). Using Lemma A.6 above, e) can be proved exactly as d). The continuous differentiability of φ in its second argument claimed in f) follows from results in [18, Sect. 18], together with similar arguments as used in the proof of Lemma A.6.

We prove that a solution u of (A.5) actually also lies in $C(\dot{J}, X_1)$, as claimed in the first statement of the theorem. Fix some $\varepsilon \in \dot{J}$. Then $u(\varepsilon) \in X_1$ and Corollary A.2a) gives $u \in C^{\beta-\alpha}([\varepsilon, T_+(u_0)), X_\alpha)$ for $\beta \in (\alpha, 1)$. Therefore $f(u(\cdot))$ as a map from $[\varepsilon, T_+(u_0))$ to X_0 is Hölder continuous and Corollary A.2b) gives $u \in C([\varepsilon, T_+(u_0)), X_1)$. Letting $\varepsilon \to 0$ the claim follows.

g) Fix some $T_1 \in (\varepsilon_2, T)$. We may assume that $\sigma(A) \subseteq [\mathrm{Re}\, z > 0]$. Otherwise add ωu to both sides of the differential equation in (A.5), where $-\omega < \inf\{ \mathrm{Re}\, z \mid z \in \sigma(A) \}$, and replace $f(u)$ by $f(u)+\omega u$. As a consequence the norms below do not depend on T_1. Fix some $u_0 \in V$ and put $u(t) := \varphi(t, u_0)$ for $t \in J(u_0)$. Also choose some fixed $\varepsilon \in (\varepsilon_1, \varepsilon_2)$ and $\beta \in (\alpha, 1)$. Applying Lemma A.1a) yields that $u \in C([\varepsilon, T_1], X_\beta)$, and the norm is independent of u_0 because $M(\varepsilon_1) \subset X_\alpha$ is bounded. Again by Lemma A.1a), $u \in C^{\beta-\alpha}([\varepsilon, T_1], X_\alpha)$. Now Lemma A.1b) gives $u \in C([\varepsilon_2, T_1], X_1)$ with a norm independent of u_0. Letting $T_1 \to T$ proves the claim. □

A.3. Asymptotics of perturbed linear equations

For our applications it is crucial to have exact knowledge of the convergence rate and the direction of solutions converging to an equilibrium. Therefore we consider the original equation as a perturbation of the linearization at an equilibrium. The statement of our results is mainly inspired by [14, Appendix B], but we have also proved a theorem for the case of continuous time dynamics. As the proof of [26, Thm. 2], which is the basis for these results, is sketchy at a central point, for convenience of the reader we give some more detail (see the proof of Theorem A.9). The proof of Theorem A.10 uses ideas from the proof of the corollary to [26, Thm. 2]. Corollary A.11 is a strengthened version of some results in [14, Appendix B].

We start with a technical Lemma.

A.7 Lemma. *Let* $(\rho_n) \subset \mathbb{R}_0^+ \cup \{\infty\}$ *and* $(\kappa_n) \subset \mathbb{R}_0^+$ *be sequences with* $\kappa_n \to 0$ *as* $n \to \infty$*,* *and let* $a, b > 0$. *Put*

$$\tilde{\rho}_n := \begin{cases} \dfrac{b\rho_n - \kappa_n(1+\rho_n)}{a + \kappa_n(1+\rho_n)} & \text{if } \rho_n < \infty \\ \dfrac{b - \kappa_n}{\kappa_n} & \text{if } \rho_n = \infty,\ \kappa_n > 0 \\ \infty & \text{if } \rho_n = \infty,\ \kappa_n = 0 \ . \end{cases}$$

a) *It holds that* $\liminf_{n\to\infty} \tilde{\rho}_n \geq \frac{b}{a} \liminf_{n\to\infty} \rho_n$.

b) *If* $b > a$ *and* $\rho_{n+1} \geq \tilde{\rho}_n$ *for all* $n \in \mathbb{N}$, *then* $\lim_{n\to\infty} \rho_n$ *exists and either* $\lim_{n\to\infty} \rho_n = 0$ *or* $\lim_{n\to\infty} \rho_n = \infty$.

Proof. a) In the case $\liminf_{n\to\infty} \rho_n = 0$ the claim is trivially true. In the case $\liminf_{n\to\infty} \rho_n > 0$ it suffices to prove that if $\lim_{n\to\infty} \rho_n \in \mathbb{R}^+ \cup \{\infty\}$ exists, then

$$\liminf_{n\to\infty} \tilde{\rho}_n \geq \frac{b}{a} \lim_{n\to\infty} \rho_n \ .$$

For $\lim_{n\to\infty} \rho_n = \infty$ the definition of $\tilde{\rho}_n$ implies $\lim_{n\to\infty} \tilde{\rho}_n = \infty$. If $\lim_{n\to\infty} \rho_n \in \mathbb{R}^+$ then

$$\liminf_{n\to\infty} \tilde{\rho}_n = \liminf_{n\to\infty} \frac{b - \kappa_n(1 + 1/\rho_n)}{a/\rho_n + \kappa_n(1+1/\rho_n)} = \frac{b}{a} \lim_{n\to\infty} \rho_n \ .$$

b) Suppose that $\limsup_{n\to\infty} \rho_n > 0$. There is a subsequence (ρ_{n_k}) of (ρ_n) with $\limsup_{n\to\infty} \rho_n = \lim_{k\to\infty} \rho_{n_k} > 0$. Applying a) to this subsequence we find

$$\lim_{k\to\infty} \rho_{n_k} \geq \liminf_{k\to\infty} \rho_{n_k+1} \geq \liminf_{k\to\infty} \tilde{\rho}_{n_k} \geq \frac{b}{a} \lim_{k\to\infty} \rho_{n_k} \ .$$

Hence $\limsup_{n\to\infty} \rho_n = \infty$.

We finish the proof by showing that for every $C \geq 0$ there is n_0 such that if $n \geq n_0$ and $\rho_n \geq C$ it follows that $\rho_{n+1} \geq C$. To see that this claims holds, assume that $\rho_n \geq C > 0$. Then

$$\rho_{n+1} \geq \tilde{\rho}_n = \frac{b - \kappa_n(1+1/\rho_n)}{a/\rho_n + \kappa_n(1+1/\rho_n)} \geq \frac{b - \kappa_n(1+1/C)}{a/C + \kappa_n(1+1/C)} \geq C$$

if n is large enough. □

We also need the following facts which are easy to prove.

A.8 Lemma. *Suppose* (s_n) *and* (t_n) *are sequences in* $\mathbb{R}^+$ *such that* $t_n \to \infty$. *For every* $a > 0$ *the following hold:*

a) $\limsup_{n\to\infty} s_n^{1/t_n} \leq a$ *if and only if* $\lim_{n\to\infty} s_n \gamma^{-t_n} = 0$ *for all* $\gamma > a$.

b) $\liminf_{n\to\infty} s_n^{1/t_n} \geq a$ *if and only if* $\lim_{n\to\infty} s_n \gamma^{-t_n} = \infty$ *for all* $\gamma \in (0, a)$.

A.3.1. Discrete dynamics

Let X be a Banach space and $T \in \mathcal{L}(X)$. Define the compact set $\Lambda := \{ |\lambda| \mid \lambda \in \sigma(T) \}$. For $\gamma \in \mathbb{R}_0^+ \setminus \Lambda$ let $P^+(\gamma)$, $P^-(\gamma)$ denote the projections in X corresponding to the spectral sets $\sigma(T) \cap [|z| < \gamma]$ and $\sigma(T) \cap [|z| > \gamma]$ respectively. The maps $\gamma \mapsto P^{\pm}(\gamma)$ are locally constant on $\mathbb{R}_0^+ \setminus \Lambda$.

A.9 Theorem. *Let $(x_n) \subseteq X \setminus \{0\}$ be a sequence satisfying $\|x_{n+1} - Tx_n\| = o(\|x_n\|)$ as $n \to \infty$, and let (a, b) be a bounded component of $\mathbb{R}^+ \setminus \Lambda$. Then one of the following alternatives holds:*

(i) *For every $\gamma \in (a, b)$*

$$\lim_{n\to\infty} \frac{\|P^-(\gamma)x_n\|}{\|P^+(\gamma)x_n\|} = \lim_{n\to\infty} \|x_n\|\gamma^{-n} = \infty$$

and $\liminf\limits_{n\to\infty} \|x_n\|^{1/n} \geq b$.

(ii) *For every $\gamma \in (a, b)$*

$$\lim_{n\to\infty} \frac{\|P^-(\gamma)x_n\|}{\|P^+(\gamma)x_n\|} = \lim_{n\to\infty} \|x_n\|\gamma^{-n} = 0$$

and $\limsup\limits_{n\to\infty} \|x_n\|^{1/n} \leq a$.

Moreover,

$$\min \Lambda \leq \liminf_{n\to\infty} \|x_n\|^{1/n} \leq \limsup_{n\to\infty} \|x_n\|^{1/n} \leq \max \Lambda \ .$$

Proof. Fix $\gamma \in (a, b)$, set $X^{\pm} := P^{\pm}(\gamma)X$ and $x^{\pm} := P^{\pm}(\gamma)x$ for $x \in X$. By choosing an equivalent norm in X we may assume that

$$\begin{aligned} \|Tx\| &\geq b_1\|x\| \qquad && x \in X^- \\ \|Tx\| &\leq a_1\|x\| \qquad && x \in X^+ \end{aligned}$$

where $a < a_1 < \gamma < b_1 < b$. Put $y_n := x_{n+1} - Tx_n$ and $\kappa_n := \|y_n\|/\|x_n\| \to 0$ as $n \to \infty$. Fix $C \geq 0$ such that $\|P^{\pm}(\gamma)\| \leq C$. Then

$$\begin{aligned} x_{n+1}^- &= Tx_n^- + y_n^- \\ x_{n+1}^+ &= Tx_n^+ + y_n^+ \end{aligned}$$

so that

$$\begin{aligned} \|x_{n+1}^-\| &\geq b_1\|x_n^-\| - C\kappa_n(\|x_n^-\| + \|x_n^+\|) \\ \|x_{n+1}^+\| &\leq a_1\|x_n^+\| + C\kappa_n(\|x_n^-\| + \|x_n^+\|) \ . \end{aligned} \tag{A.9}$$

Define

$$\rho_n := \begin{cases} \dfrac{\|x_n^-\|}{\|x_n^+\|} & \text{if } \|x_n^+\| > 0 \\ \infty & \text{if } \|x_n^+\| = 0 \,. \end{cases}$$

Then

$$\rho_{n+1} \ge \tilde{\rho}_n := \begin{cases} \dfrac{b_1\rho_n - C\kappa_n(1+\rho_n)}{a_1 + C\kappa_n(1+\rho_n)} & \text{if } \rho_n < \infty \\ \dfrac{b_1 - C\kappa_n}{C\kappa_n} & \text{if } \rho_n = \infty,\ \kappa_n > 0 \\ \infty & \text{if } \rho_n = \infty,\ \kappa_n = 0 \end{cases}$$

By Lemma A.7 either $\rho_n \to \infty$ or $\rho_n \to 0$. In the first case, for large n

$$\frac{1}{2}\|x_n^-\| \le \|x_n\| \le 2\|x_n^-\| \,.$$

Together with (A.9) we find $b_2 \in (\gamma, b_1)$ with $\|x_{n+1}^-\| \ge b_2\|x_n^-\|$ for large n. With some large n_0 it follows that

$$\|x_n\|\gamma^{-n} \ge \frac{1}{2}\|x_n^-\|\gamma^{-n} \ge \frac{1}{2}\|x_{n_0}^-\|\left(\frac{b_2}{\gamma}\right)^{n-n_0}\gamma^{-n_0} \to \infty$$

as $n \to \infty$. In the case that $\rho_n \to 0$, for large n

$$\frac{1}{2}\|x_n^+\| \le \|x_n\| \le 2\|x_n^+\| \,.$$

By (A.9) there is $a_2 \in (a_1, \gamma)$ with $\|x_{n+1}^+\| \le a_2\|x_n^+\|$ for large n. Similarly as above it follows that $\|x_n\|\gamma^{-n} \to 0$ as $n \to \infty$.

The alternative is independent of $\gamma \in (a, b)$ since $P^{\pm}(\gamma)$ is independent. The other statements follow easily from Lemma A.8 and the considerations above. □

A.3.2. Continuous time dynamics

In this section we study the behavior of the semiflow φ given by (A.5) near an equilibrium point via linearization. We suppose in addition to the hypotheses of Section A.2 that $f \in C^1(X_\alpha, X_0)$ uniformly on bounded subsets and $f(0) = 0$.

Set $L := A - f'(0)$ and $g(u) := f(u) - f'(0)u$ for $u \in X_\alpha$. Then for every $\varepsilon > 0$ there is $C(\varepsilon) \ge 0$ with

$$\|f'(0)u\|_0 \le \|f'(0)\|_{\alpha,0}\|u\|_\alpha \le \varepsilon\|u\|_1 + C(\varepsilon)\|u\|_0$$

for $u \in X_1$, by interpolation inequalities and Young's inequality. Hence by [3, Theorem I.1.3.1] $L \in \mathcal{H}(X_1, X_0)$. The problem

$$\text{(A.10)} \qquad \begin{aligned} \dot{u}(t) + Lu(t) &= g(u(t)) & & t > 0 \\ u(0) &= u_0 & & u_0 \in X_\alpha \end{aligned}$$

is equivalent with (A.5), and moreover $g'(0) = 0$.

If $u_1, u_2 \in C(\mathbb{R}_0^+, X_\alpha)$ are solutions of (A.10) with $u_i(t) \to 0$ as $t \to \infty$ $(i = 1, 2)$, define $B \in C(\mathbb{R}_0^+, \mathcal{L}(X_\alpha, X_0))$ by

$$B(t) := \int_0^1 g'(su_1(t) + (1 - s)u_2(t))\, ds\ . \tag{A.11}$$

Then $v := u_1 - u_2$ is a solution of

$$\dot{v}(t) + Lv(t) = B(t)v(t)\ . \tag{A.12}$$

Similarly, if $u \in C(\mathbb{R}_0^+, X_\alpha)$ is a solution of (A.10) with $u(t) \to 0$ as $t \to \infty$, define $B \in C(\mathbb{R}_0^+, \mathcal{L}(X_\alpha, X_0))$ by

$$B(t) := g'(u(t))\ . \tag{A.13}$$

Then $v(t) := D\varphi^t(u(0))v_0$ is, for $v_0 \in X_\alpha$, a mild solution of (A.12) by Theorem A.3f). In any case, $\|B(t)\|_{\alpha,0} \to 0$ as $t \to \infty$.

In order to state the results about the asymptotic behavior we set

$$\Lambda := \{\, \mathrm{Re}\, \lambda \mid \lambda \in \sigma(L)\, \}\ .$$

Since L is sectorial, Λ is closed in $\mathbb{R}$. For $\gamma \in \mathbb{R} \setminus \Lambda$ let $P^+(\gamma)$, $P^-(\gamma)$ denote the projections in X_0 corresponding to the spectral sets $\sigma(L) \cap [\mathrm{Re}\, z > \gamma]$ and $\sigma(L) \cap [\mathrm{Re}\, z < \gamma]$. Clearly the maps $\gamma \mapsto P^\pm(\gamma)$ are locally constant on $\mathbb{R} \setminus \Lambda$.

Suppose now that $B \in C(\mathbb{R}_0^+, \mathcal{L}(X_\alpha, X_0))$ satisfies $\|B(t)\|_{\alpha,0} \to 0$ and that $v \in C(\mathbb{R}_0^+, X_\alpha)$ is a mild solution of (A.12) with $v(t) \neq 0$ for $t \geq 0$. With arguments similar to those in the proof of Lemma A.1 one can show that $v(t) \in X_\beta$ for all $\beta \in [0, 1)$ and $t > 0$.

A.10 Theorem. *If (a, b) is a bounded component of $\mathbb{R} \setminus \Lambda$, then one of the following alternatives holds:*

(i) *For every $\beta \in [0, 1)$, $\gamma \in (a, b)$:*

$$\lim_{t\to\infty} \frac{\|P^-(\gamma)v(t)\|_\beta}{\|P^+(\gamma)v(t)\|_\beta} = \lim_{t\to\infty} \|v(t)\|_\beta e^{\gamma t} = \infty$$

and $\liminf_{t\to\infty} \|v(t)\|_\beta^{1/t} \geq e^{-a}$.

(ii) *For every $\beta \in [0, 1)$, $\gamma \in (a, b)$:*

$$\lim_{t\to\infty} \frac{\|P^-(\gamma)v(t)\|_\beta}{\|P^+(\gamma)v(t)\|_\beta} = \lim_{t\to\infty} \|v(t)\|_\beta e^{\gamma t} = 0$$

and $\limsup_{t\to\infty} \|v(t)\|_\beta^{1/t} \leq e^{-b}$.

Moreover,

$$e^{-\sup\Lambda} \le \liminf_{t\to\infty} \|v(t)\|_\beta^{1/t} \le \limsup_{t\to\infty} \|v(t)\|_\beta^{1/t} \le e^{-\min\Lambda}$$

where $e^{-\infty} = 0$ *is to be understood.*

Proof. Put $P^\pm := P^\pm(\gamma)$ for any $\gamma \in (a,b)$. Also set $x^\pm := P^\pm x$ for $x \in X_0$. For $r \ge 0$ and $t \in [0,1]$ we define $B_r(t) := B(r+t)$. By [14, Appendix A], for each $r \ge 0$ and $s \in [0,1)$ the Cauchy problem

$$\text{(A.14)} \qquad \begin{cases} \dot{w} + Lw = B_r(t)w & s < t \le 1 \\ w(s) = w_0 & w_0 \in X_0 \end{cases}$$

has a mild solution $w \in C([s,1], X_0) \cap L_1((s,1), X_\alpha)$. For notational convenience we introduce the corresponding "evolution operator" $U_r(t,s)$ ($0 \le s \le t \le 1$) by defining $U_r(t,s)w_0 := w(t)$ where w is the mild solution of (A.14). Then

$$\text{(A.15)} \qquad U_r(t,s)v(r+s) = v(r+t)\ .$$

Put $V(t,s) := e^{-L(t-s)}$ for $t \ge s$. By [14, Theorem A.1] there are constants $C_0, C_1 > 0$ such that

$$\text{(A.16)} \qquad \kappa_\beta(r) := \sup_{t\in[0,1]} \|V(t,0) - U_r(t,0)\|_{\beta,\beta} \le C_0 \sup_{t\in[0,1]} \|B_r(t)\|_{\alpha,0}$$

and

$$\text{(A.17)} \qquad \|U_r(1,0)\|_{\beta',\beta} \le \|V(1,0)\|_{\beta',\beta} + \|V(1,0) - U_r(1,0)\|_{\beta',\beta} \le C_1$$

hold for all $\beta, \beta' \in [0,1)$ and $r \ge 0$. Here C_0 and C_1 depend on L, β, β', α, but not on r. It follows that $\kappa_\beta(r) \to 0$ as $r \to \infty$.

For $n \in \mathbb{N}_0$ and $\beta \in [0,1)$ define $T, T_n \in \mathcal{L}(X_\beta)$ by $T := V(1,0)$ and $T_n := U_n(1,0)$, so that

$$\|T - T_n\|_{\beta,\beta} \le \kappa_\beta(n) \to 0$$

as $n \to \infty$. As a consequence of (A.15) we have

$$v(n+1) = T_n v(n) = Tv(n) + (T_n - T)v(n)\ .$$

Moreover, the spectral mapping theorem [31, Cor. 2.3.7] yields

$$\{\, |\lambda| \mid \lambda \in \sigma(T)\,\} \setminus \{0\} = \{\, e^{-\lambda} \mid \lambda \in \Lambda\,\}\ .$$

For $t \ge 0$ put

$$\rho_\beta(t) := \begin{cases} \dfrac{\|v^-(t)\|_\beta}{\|v^+(t)\|_\beta} & \text{if } v^+(t) \ne 0 \\ \infty & \text{if } v^+(t) = 0. \end{cases}$$

By Theorem A.9 exactly one of the following alternatives applies:

(i′) For every $\gamma \in (a, b)$

$$\lim_{n\to\infty} \|v(n)\|_\beta e^{\gamma n} = \lim_{n\to\infty} \rho_\beta(n) = \infty$$

(ii′) For every $\gamma \in (a, b)$

$$\lim_{n\to\infty} \|v(n)\|_\beta e^{\gamma n} = \lim_{n\to\infty} \rho_\beta(n) = 0$$

Suppose for some $\beta \in [0, 1)$ alternative (i′) above holds true. Then for every $\beta' \in [0, 1)$, (A.17) implies

$$\|v(n+1)\|_\beta = \|T_n v(n)\|_\beta \leq \|T_n\|_{\beta',\beta} \|v(n)\|_{\beta'} \leq C_1 \|v(n)\|_{\beta'}$$

where C_1 is independent of n. Thus

$$\|v(n)\|_{\beta'} e^{\gamma n} \geq \frac{1}{C_1} \|v(n+1)\|_\beta e^{\gamma(n+1)} e^{-\gamma} \to \infty$$

for all $\gamma \in (a, b)$ as $n \to \infty$, so that alternative (i′) holds for β'. If alternative (ii′) holds for β, exchanging the rôles of β and β' in the argument above, we see that (ii′) also holds for β'. This shows that the validity of alternative (i′) or (ii′) is independent of $\beta \in [0, 1)$.

Now fix any $\beta \in [0, 1)$ and $\gamma \in (a, b)$. There are constants $C_2, C_3 > 0$ such that

$$\begin{aligned} &\|V(s,0)x\|_\beta \geq C_2 \|x\|_\beta \qquad x \in P^- X_\beta \\ &\|V(s,0)x\|_\beta \leq C_3 \|x\|_\beta \qquad x \in P^+ X_\beta \ . \end{aligned} \tag{A.18}$$

holds for all $s \in [0, 1]$. Fix C_4 with $\|P^\pm\|_{\beta,\beta} \leq C_4$. For every $s \in [0, 1]$ and every $t \geq 0$ we have from (A.15)

$$\begin{aligned} v^-(t+s) &= V(s,0)v^-(t) + P^-(U_t(s,0) - V(s,0))v(t) \\ v^+(t+s) &= V(s,0)v^+(t) + P^+(U_t(s,0) - V(s,0))v(t) \end{aligned}$$

so that

$$\begin{aligned} \|v^-(t+s)\|_\beta &\geq C_2 \|v^-(t)\|_\beta - C_4 \kappa_\beta(t)(\|v^-(t)\|_\beta + \|v^+(t)\|_\beta) \\ \|v^+(t+s)\|_\beta &\leq C_3 \|v^+(t)\|_\beta + C_4 \kappa_\beta(t)(\|v^-(t)\|_\beta + \|v^+(t)\|_\beta) \ . \end{aligned}$$

Setting

$$\tilde{\rho}(t) := \begin{cases} \dfrac{C_2 \rho_\beta(t) - C_4 \kappa_\beta(t)(1 + \rho_\beta(t))}{C_3 + C_4 \kappa_\beta(t)(1 + \rho_\beta(t))} & \text{if } \rho_\beta(t) < \infty \\[2ex] \dfrac{C_2 - C_4 \kappa_\beta(t)}{C_4 \kappa_\beta(t)} & \text{if } \rho_\beta(t) = \infty,\ \kappa_\beta(t) > 0 \\[2ex] \infty & \text{if } \rho_\beta(t) = \infty,\ \kappa_\beta(t) = 0 \end{cases}$$

it follows that

$$\text{(A.19)} \qquad \rho_\beta(t+s) \ge \tilde{\rho}(t) \qquad \text{for } t \ge 0,\, s \in [0,1].$$

If $\rho_\beta(t) \to 0$ as $t \to \infty$, alternative (ii′) holds. For $s \in [0,1]$ and $n \in \mathbb{N}$

$$\|v(n+s)\|_\beta \le \|V(s,0)v(n)\|_\beta + \kappa_\beta(n)\|v(n)\|_\beta \le C\|v(n)\|_\beta$$

and therefore

$$\|v(t)\|_\beta e^{\gamma t} \le C\|v([t])\|_\beta e^{\gamma [t]} e^{\gamma (t-[t])} \to 0$$

as $t \to \infty$. Since $\gamma \in (a,b)$ is chosen arbitrarily, Lemma A.8 gives $\limsup_{t\to\infty}\|v(t)\|_\beta^{1/t} \le e^{-b}$.

Now suppose that $\rho_\beta(t) \not\to 0$. There is a sequence $t_k \to \infty$ such that $\lim_{k\to\infty}\rho_\beta(t_k) > 0$. By Lemma A.7 we have $\liminf_{k\to\infty}\tilde{\rho}(t_k) > 0$. From (A.19) we find a sequence $(n_k) \subseteq \mathbb{N}$ with $\lim_{k\to\infty}\rho_\beta(n_k) > 0$. Thus alternative (i′) must hold, i.e. $\rho_\beta(n) \to \infty$ as $n \to \infty$. By Lemma A.7 again $\tilde{\rho}(n) \to \infty$ and by (A.19) again $\rho_\beta(t) \to \infty$ as $t \to \infty$.

We need to show the remaining statements about the asymptotics for alternative (i). For $n \in \mathbb{N}$ and $s \in [0,1]$ we have with (A.15) and (A.18)

$$\begin{aligned}
\|v(n+s)\|_\beta &\ge C_2\|v^-(n)\|_\beta - C_3\|v^+(n)\|_\beta - \kappa_\beta(n)\|v(n)\|_\beta \\
&= \left(\frac{C_2\rho_\beta(n) - C_3}{\|v(n)\|_\beta/\|v^+(n)\|_\beta} - \kappa_\beta(n)\right)\|v(n)\|_\beta \\
&\ge \left(\frac{C_2\rho_\beta(n) - C_3}{\rho_\beta(n)+1} - \kappa_\beta(n)\right)\|v(n)\|_\beta \\
&\ge \frac{C_2}{2}\|v(n)\|_\beta
\end{aligned}$$

for n large, since $\rho_\beta(n) \to \infty$. Thus

$$\|v(t)\|_\beta e^{\gamma t} \ge \frac{C_2}{2}\|v([t])\|_\beta e^{\gamma [t]} e^{\gamma (t-[t])} \to \infty$$

as $t \to \infty$ since (i′) holds. Again, $\gamma \in (a,b)$ was arbitrary, so that by Lemma A.8 $\liminf_{t\to\infty}\|v(t)\|_\beta^{1/t} \ge e^{-a}$.

The remaining assertions are simple consequences of the above considerations and of Theorem A.9. □

We can now set $\chi(\gamma) := \lim_{t\to\infty}\|v(t)\|_0 e^{\gamma t}$ for $\gamma \in \mathbb{R} \setminus \Lambda$. Then χ is locally constant on $\mathbb{R} \setminus \Lambda$ and nondecreasing. Moreover, for every $\beta \in [0,1)$ we have $\chi(\gamma) = \lim_{t\to\infty}\|v(t)\|_\beta e^{\gamma t}$. If $\chi \equiv 0$, then $\lim_{t\to\infty}\|v(t)\|_\beta^{1/t} = 0$ for all $\beta \in [0,1)$.

A.11 Corollary. *Suppose that* $\chi \not\equiv 0$.

a) *There is* $\lambda \in \Lambda$ *such that*

$$\begin{aligned}
\chi(\gamma) &= 0 \qquad && \textit{if } \gamma < \lambda \\
\chi(\gamma) &= \infty \qquad && \textit{if } \gamma > \lambda
\end{aligned}$$

for all $\gamma \in \mathbb{R} \setminus \Lambda$*. For this* λ *the following hold:*

b) *If*
$$\lambda \in \overline{(-\infty, \lambda) \setminus \Lambda} \cap \overline{(\lambda, \infty) \setminus \Lambda} \, ,$$
then $\lim_{t\to\infty} \|v(t)\|_\beta^{1/t} = e^{-\lambda}$ *for all* $\beta \in [0, 1)$.

c) *For* $a, b \in \mathbb{R} \setminus \Lambda$ *with* $a < \lambda < b$ *set* $P_* := P^+(a) - P^+(b)$*. Note that* P_* *is the projection corresponding to the spectral set* $\sigma(L) \cap [\mathrm{Re}\, z \in (a, b)]$*. For* $\beta \in [0, 1)$ *define*
$$S_{1,\beta} := \{ x \in P_* X_\beta \mid \|x\|_\beta = 1 \} \, .$$
Then $\mathrm{dist}_{X_\beta}(v(t)/\|v(t)\|_\beta, S_{1,\beta}) \to 0$ *as* $t \to \infty$*. Moreover, if* $\beta' \in [\beta, 1)$ *and*
$$v(t)/\|v(t)\|_{\beta'} \to v_1 \in S_{1,\beta'} = \{ x \in P_* X_{\beta'} \mid \|x\|_{\beta'} = 1 \}$$
in $X_{\beta'}$*, then* $v(t)/\|v(t)\|_\beta \to v_1/\|v_1\|_\beta$ *in* X_β*.*

Proof. Statements a) and b) are obvious consequences of the properties of Λ and χ. To prove c), fix some $\beta \in [0, 1)$ and put $Q_* := I - P_*$. By Theorem A.10 we have
$$\frac{\|P^-(a)v(t)\|_\beta}{\|P^+(a)v(t)\|_\beta} \to 0 \, , \qquad \frac{\|P^-(b)v(t)\|_\beta}{\|P^+(b)v(t)\|_\beta} \to \infty$$
as $t \to \infty$. Thus
$$\frac{\|P^-(a)v(t)\|_\beta}{\|v(t)\|_\beta} \le \frac{\|P^-(a)v(t)\|_\beta}{\|P^+(a)v(t)\|_\beta - \|P^-(a)v(t)\|_\beta} \to 0$$
and
$$\frac{\|P^+(b)v(t)\|_\beta}{\|v(t)\|_\beta} \le \frac{\|P^+(b)v(t)\|_\beta}{\|P^-(b)v(t)\|_\beta - \|P^+(b)v(t)\|_\beta} \to 0$$
Therefore, from
$$v(t) = P^-(a)v(t) + P^+(b)v(t) + P_* v(t)$$
it follows that
$$\frac{\|P_* v(t)\|_\beta}{\|v(t)\|_\beta} \to 1 \qquad \text{and} \qquad \frac{\|Q_* v(t)\|_\beta}{\|v(t)\|_\beta} \to 0 \, .$$
Now set
$$x(t) := \frac{P_* v(t)}{\|v(t)\|_\beta} \qquad \text{and} \qquad y(t) := \frac{Q_* v(t)}{\|v(t)\|_\beta} \, .$$
Then $x(t)/\|x(t)\|_\beta \in S_{1,\beta}$ and
$$\left\| \frac{x(t)}{\|x(t)\|_\beta} - \frac{v(t)}{\|v(t)\|_\beta} \right\|_\beta \le \left\| \frac{x(t)}{\|x(t)\|_\beta} - x(t) \right\|_\beta + \|y(t)\|_\beta$$
$$= \left|1 - \|x(t)\|_\beta\right| + \|y(t)\|_\beta \to 0 \, .$$

Hence

$$\operatorname{dist}_{X_\beta}\left(\frac{v(t)}{\|v(t)\|_\beta}, S_{1,\beta}\right) \to 0$$

as $t \to \infty$.

To prove the last statement, suppose that $v(t)/\|v(t)\|_{\beta'} \to v_1 \in S_{1,\beta'}$ in $X_{\beta'}$. This convergence is also true in X_β. Hence $\|v(t)\|_\beta/\|v(t)\|_{\beta'} \to \|v_1\|_\beta$ and

$$\frac{v(t)}{\|v(t)\|_\beta} = \frac{v(t)}{\|v(t)\|_{\beta'}} \frac{\|v(t)\|_{\beta'}}{\|v(t)\|_\beta} \to \frac{v_1}{\|v_1\|_\beta}$$

in X_β as $t \to \infty$. □

A.4. Superstable manifolds

In this section we construct submanifolds of the strong stable manifold of 0, if 0 is an equilibrium point of φ. Therefore recall the situation from Section A.3.2. We suppose that (a, b) with $b > 0$ is a bounded connected component of $\mathbb{R} \setminus \Lambda$. Fix $\gamma \in (\max\{0, a\}, b)$ and put $P^\pm := P^\pm(\gamma)$ and $X^\pm_\alpha := P^\pm X_\alpha$. The next lemma is based on [9, Lem. 4.1]. We prove some additional facts, in particular we give a classification of tangent vectors. We use the notions of invariance introduced in Section 2 and also set $\mathcal{I}_+ := T_+^{-1}(\infty)$.

A.12 Theorem. *There are $M \geq 1$ and $\rho, \eta > 0$ such that defining*

$$W_{\text{loc}} := \{\, u \in \mathcal{I}_+ \mid \|P^+ u\|_\alpha < \eta,\ \sup_{t\geq 0} \|\varphi(t, u)\|_\alpha e^{\gamma t} \leq \rho \,\} ,$$

the following holds:

a) *W_{loc} is a C^1-submanifold of X_α such that $T_0 W_{\text{loc}} = X^+_\alpha$, and W_{loc} is C^1-diffeomorphic to $U_\eta X^+_\alpha$ under the restriction of P^+ to W_{loc}.*

b) *W_{loc} is locally invariant with respect to φ.*

c) *For every $r > 0$ there is $t \geq 0$ such that $\varphi([t, \infty), W_{\text{loc}}) \subseteq W_{\text{loc}} \cap U_r X_\alpha$.*

Consider $u_0 \in W_{\text{loc}}$, $u(t) := \varphi(t, u_0)$, $v_0 \in X_\alpha$, and $v(t) := D\varphi^t(u_0)v_0$.

d) *If $v_0 \in T_{u_0} W_{\text{loc}}$, the tangent space of W_{loc} at u_0, then $v(t) \in T_{u(t)} W_{\text{loc}}$ for $t \geq 0$ and*

$$\sup_{t\geq 0} \|v(t)\|_\alpha e^{\gamma t} \leq 2M \|P^+ v_0\|_\alpha \ . \tag{A.20}$$

e) *If*

$$\sup_{t\geq 0} \|v(t)\|_\alpha e^{\gamma t} < \infty \ ,$$

then $v(t) \in T_{u(t)} W_{\text{loc}}$ for $t \geq 0$ and (A.20) *holds.*

Proof. Parts of this theorem have been proved in [9, Lem. 4.1]. For better reference we sketch their arguments and show how they can be extended to prove our claims.

Let $U^{\pm}(t,s) := \exp(-P^{\pm}L(t-s))$ denote the evolution operator generated by $-P^{\pm}L$ in $X_0^{\pm}$. Here $U^{+}(t,s)$ is defined for $t \geq s$, and $U^{-}(t,s)$ is defined for $s,t \in \mathbb{R}$ since $P^{-}L \in \mathcal{L}(X_0^{-})$. For $u \in X_0$ we write $u^{\pm} := P^{\pm}u$. For notational convenience we write $\gamma_1 := \gamma$, and we pick some $\gamma_2 \in (\gamma_1, b)$, $\beta \in (\max\{0,a\}, \gamma_1)$ and $\delta \in (\gamma_2, b)$.

There is $M \geq 1$, depending only on L, β and δ, such that

$$\begin{aligned} \|U^{+}(t,0)\|_{\alpha,\alpha} &\leq Me^{-\delta t} && t \geq 0 \\ \|U^{+}(t,0)\|_{0,\alpha} &\leq Mt^{-\alpha}e^{-\delta t} && t \geq 0 \\ \|U^{-}(t,0)\|_{0,\alpha} &\leq Me^{-\beta t} && t \leq 0 . \end{aligned}$$

These operator norms are to be understood for the respective restrictions to $X^{\pm}$.

We introduce the Banach spaces

$$V_i := \{ v \in C([0,\infty), X_\alpha) \mid \sup_{t\geq 0} \|v(t)\|_\alpha e^{\gamma_i t} < \infty \}$$

with norms $\|v\|_{V_i} := \sup_{t\geq 0}\|v(t)\|_\alpha e^{\gamma_i t}$ for $i = 1,2$. If $x \in X_\alpha^{+}$ and $u \in L_\infty((0,\infty), X_\alpha)$ define $F_x(u) \in C([0,\infty), X_\alpha)$ by

$$F_x(u)(t) := U^{+}(t,0)x + \int_0^t U^{+}(t,s)P^{+}g(u(s))\,ds - \int_t^\infty U^{-}(t,s)P^{-}g(u(s))\,ds \ .$$

For $i = 1,2$ it follows as in [9] that if $u \in V_i$, then u is a solution of (A.10) if and only if $F_x(u) = u$ with $x = P^{+}u_0$.

Put

$$k(\rho) := \sup_{u \in B_\rho X_\alpha} \|g'(u)\|_{\alpha,0} \ .$$

Then $k(\rho) \to 0$ as $\rho \to 0$. Also set

$$C := \max_{i=1,2}\left(\|P^{+}\|_{0,0} \int_0^\infty t^{-\alpha} e^{(\gamma_i - \delta)t}\,dt + \|P^{-}\|_{0,0} \int_0^\infty e^{(\beta-\gamma_i)t}\,dt \right) .$$

Now we choose $\rho > 0$ small enough such that $k(\rho)MC \leq 1/2$. Estimates as in [9] show that then for every $x \in B_{\rho/2M}X^{+}$ and $i = 1,2$ the map $F_x\colon B_\rho V_i \to B_\rho V_i$ is a contraction. The arguments in [9] show that the sets

$$W_i := \{ u \in \mathcal{I}_{+} \mid \|P^{+}u\|_\alpha < \rho/2M,\ \sup_{t\geq 0}\|\varphi(t,u)\|_\alpha e^{\gamma_i t} \leq \rho \}$$

are C^1-submanifolds of X_α, given as local graphs of maps $U_{\rho/2M}X_\alpha^{+} \to X_\alpha^{-}$, such that $T_0 W_i = X_\alpha^{+}$. Since $W_2 \subseteq W_1$ and the W_i are graphs over the same base set, we actually have

$$W_1 = W_2 \ . \tag{A.21}$$

We now choose $\eta \in (0, \rho/2M]$ small enough such that W_{loc} as defined in the statement of the lemma satisfies

$$W_{\text{loc}} \subseteq U_{\rho/2} X_\alpha \ . \tag{A.22}$$

Then a) holds true.

We have

$$\sup_{s \geq 0} \|\varphi(s, u)\|_\alpha e^{\gamma_1 s} \leq \rho \quad \Longrightarrow \quad \forall t \geq 0\colon \sup_{s \geq 0} \|\varphi(s, \varphi(t, u))\|_\alpha e^{\gamma_1 s} \leq \rho \tag{A.23}$$

Therefore W_{loc} is locally positive invariant. To show local invariance for negative times, suppose we are given $u \in W_{\text{loc}}$, $(u_n) \subseteq X_\alpha$, $u_n \to u$, $t_n \geq 0$, $t_n \to 0$, with $\varphi(t_n, u_n) \to u$ and $\varphi(t_n, u_n) \in W_{\text{loc}}$. We need to show that $u_n \in W_{\text{loc}}$ for large n. Since $u \in W_{\text{loc}}$, for large n

$$\|u_n^+\| < \eta \ . \tag{A.24}$$

Pick $s_0 > 0$ such that $e^{\gamma_1 s_0} \leq 2$ and $\varphi([0, s_0], u) \subseteq U_{\rho/2} X_\alpha$. This is possible by (A.22). We claim that for large n

$$\sup_{s \geq 0} \|\varphi(s, u_n)\|_\alpha e^{\gamma_1 s} \leq \rho \ . \tag{A.25}$$

If this is not the case, extracting a subsequence we may assume that there exists $(s_n) \subseteq \mathbb{R}_0^+$ with

$$\|\varphi(s_n, u_n)\|_\alpha e^{\gamma_1 s_n} > \rho \ . \tag{A.26}$$

Moreover, we may assume that either

$$s_n \to s \in [0, s_0] \tag{A.27}$$

or

$$\forall n\colon s_n \geq s_0 \ . \tag{A.28}$$

If (A.27) is the case, then

$$\|\varphi(s_n, u_n)\|_\alpha e^{\gamma_1 s_n} \to \|\varphi(s, u)\|_\alpha e^{\gamma_1 s} < \rho \ ,$$

contradicting (A.26). If (A.28) holds, in view of (A.21), $\varphi(t_n, u_n) \in W_{\text{loc}}$, and $t_n \to 0$, we find

$$\begin{aligned}\|\varphi(s_n, u_n)\|_\alpha e^{\gamma_1 s_n} &= \|\varphi(s_n - t_n, \varphi(t_n, u_n))\|_\alpha e^{\gamma_1 s_n} \\ &\leq \rho e^{\gamma_2(-s_n + t_n) + \gamma_1 s_n} \leq \rho e^{(\gamma_1 - \gamma_2) s_0 + \gamma_2 t_n} \to \rho e^{(\gamma_1 - \gamma_2) s_0} < \rho \ .\end{aligned}$$

This also contradicts (A.26). Thus (A.25) holds for large n, and together with (A.24) it follows that $u_n \in W_{\text{loc}}$ for large n.

From these facts we conclude that if $u \in W_{\text{loc}}$ there is $r > 0$ such that if $t > 0$, $v \in X_\alpha$ with $\varphi([0,t],v) \subseteq B_r(u)X_\alpha$ and $\varphi(t,v) \in W_{\text{loc}}$, then $v \in W_{\text{loc}}$. This proves local negative invariance of W_{loc} and therefore b).

From the definition of W_{loc} it is clear that $\varphi(t,u) \to 0$ uniformly in $u \in W_{\text{loc}}$. Together with (A.23) property c) follows.

Fix $u_0 \in W_{\text{loc}}$, put $u(t) := \varphi(t,u_0)$ for $t \geq 0$, and put $x := P^+u_0 \in U_\eta X_\alpha^+$. Let h be the inverse of the restriction of P^+ to W_{loc}. Then $h(x) = u_0$. For $y \in X_\alpha^+$ consider the map $G_y \colon V_1 \to V_1$ given by

$$G_y[v](t) := U^+(t,0)y + \int_0^t U^+(t,s)P^+g'(u(s))v(s)\,ds \\ - \int_t^\infty U^-(t,s)P^-g'(u(s))v(s)\,ds \ .$$

By estimates similar to those proving that $F_x \colon B_\rho V_i \to B_\rho V_i$ is a contraction we find that

$$\|G_y v\|_{V_1} \leq M\|y\|_\alpha + Mk(\rho)C\|v\|_{V_1} \leq M\|y\|_\alpha + \frac{1}{2}\|v\|_{V_1} \tag{A.29}$$

and

$$\|G_y[v_1 - v_2]\|_{V_1} \leq Mk(\rho)C\|v_1 - v_2\|_{V_1} \leq \frac{1}{2}\|v_1 - v_2\|_{V_1} \ . \tag{A.30}$$

Therefore G_y has a unique fixed point $v_y \in V_1$ that satisfies $\|v_y\|_{V_1} \leq 2M\|y\|_\alpha$.

Now $v \in V_1$ satisfies $v(t) = D\varphi^t(u_0)v(0)$ if and only if v is the unique fixed point of $G_{v(0)^+}$. Moreover, if $v \in V_1$ is, for some $y \in X_\alpha^+$, the unique fixed point of G_y, then $v(0) = h'(x)y$ and $v(t) \in T_{u(t)}W_{\text{loc}}$ for $t \geq 0$. This follows from the sentence containing Equation (4.6) in [9]. These observations prove d) and e). □

A.13 Remark. We have no reference for the local negative invariance of W_{loc} nor for the (global) positive invariance of W_{loc}.

The proof of the fact that $\mathcal{O}_-(W_{\text{loc}})$ is a manifold is usually based on [25, Thm. 6.1.9]. There are simple counterexamples though that render that theorem false as stated. A sufficient condition on W_{loc} in order to construct a global manifold has been proved in Theorem A.12c) above, as we will show in the next theorem.

A.14 Theorem. *Define the invariant set*

$$W := \{\, u \in \mathcal{I}_+ \mid \limsup_{t\to\infty} \|\varphi(t,u)\|_\alpha^{1/t} \leq e^{-b} \,\} \ .$$

Let W_{loc} be given by Theorem A.12*. Then*

$$W = \mathcal{O}_-(W_{\text{loc}}) \ .$$

Suppose in addition that $\dim(X_\alpha^-) < \infty$ *and that for every* $t \geq 0$ *and every* $u \in \mathcal{D}_t$ *the map* $D\varphi^t(u) \in \mathcal{L}(X_\alpha)$ *has dense range. Then*

$$\bigcup_{s\in[0,t]} \varphi^{-s}(W_{\text{loc}}) \subset W$$

is a submanifold of X_α for all $t \geq 0$, and W is an injectively immersed C^1-manifold with $T_0 W = X_\alpha^+$. If $u_0 \in W$, $v_0 \in X_\alpha$, and $v(t) := D\varphi^t(u_0)v_0$, then $v_0 \in T_{u_0}W$ if and only if

$$\limsup_{t\to\infty} \|v(t)\|_\alpha^{1/t} \leq e^{-b} .$$

Proof. Let M, ρ, η and W_{loc} be given by Theorem A.12. Put

$$\widetilde{W} := \mathcal{O}_-(W_{\mathrm{loc}}) .$$

First suppose that $u_0 \in W$ and put $u(t) := \varphi(t, u_0)$. Then

$$\limsup_{t\to\infty} \|u(t)\|_\alpha e^{\gamma t} = 0$$

by the definition of W and Lemma A.8. For t large enough we have

$$\sup_{s\geq 0} \|\varphi(s, u(t))\|_\alpha e^{\gamma s} \leq \rho$$

and $\| P^+ u(t)\|_\alpha < \eta$. Hence $u(t) \in W_{\mathrm{loc}}$ for t large and $u_0 \in \widetilde{W}$.

Now suppose that $u_0 \in \widetilde{W}$ and put $u(t) := \varphi(t, u_0)$ again. Then

$$\limsup_{t\to\infty} \|u(t)\|_\alpha e^{\gamma t} < \infty . \tag{A.31}$$

Comparing u with the zero solution, from (A.31) and Theorem A.10 we conclude that $u_0 \in W$.

Suppose now that the statement about the denseness of the range holds. Put $m := \dim(X_\alpha^-)$. For $\Sigma \subseteq \mathbb{R}_0^+$ let us denote

$$W(\Sigma) := \bigcup_{t\in\Sigma} \varphi^{-t}(W_{\mathrm{loc}}) .$$

For one-point sets $\Sigma = \{t\}$ we write $W(t) := W(\{t\})$. It is sufficient to show that for every $t \geq 0$ the set $W([0, t])$ is an m-codimensional submanifold of X_α.

By A.12c) there is $T \geq 0$ such that $\varphi([T, \infty), W_{\mathrm{loc}}) \subseteq W_{\mathrm{loc}}$. Fix some $t \geq 0$. For every $s \in [0, t]$ we have

$$\varphi(t + T, W(s)) = \varphi(t - s + T, \varphi(s, W(s))) \subseteq \varphi(t - s + T, W_{\mathrm{loc}}) \subseteq W_{\mathrm{loc}}$$

since $t - s + T \geq T$. Hence $W([0, t]) \subseteq W(t + T)$. If $u \in W([0, t])$, there is $s \in [0, t]$ with

$$u \in W(s) \subseteq W([0, t]) \subseteq W(t + T) .$$

The arguments in the proof of [25, Thm 6.1.9] show that both $W(s)$ and $W(t + T)$ are m-codimensional C^1-submanifolds. Hence there is $r > 0$ such that $U_r(u) \cap W(s) = U_r(u) \cap W(t + T)$. It follows that also $U_r(u) \cap W([0, t]) = U_r(u) \cap W(t + T)$. Since $u \in W([0, t])$ was arbitrary, $W([0, t])$ is an m-codimensional submanifold of X_α. We have proved that W is an injectively immersed manifold of codimension m. In this case the characterization of tangent vectors follows from A.12 by similar arguments as in the proof that $\widetilde{W} = W$. □

A.15 Remark. Theorem A.10 shows that the construction of the superstable manifolds is essentially independent of α. This means, using obvious notation, that if $\alpha' \in [\alpha, 1)$, then $W_{\alpha'} = X_{\alpha'} \cap W_\alpha$.

A characterization of tangent vectors similar to that given in the preceding theorems was also stated in [11, Lem. 4.b.1].

A.5. Unique continuation

In this section let (X_0, X_1) be a densely injected couple of real Hilbert spaces and assume that $A \in \mathcal{H}(X_1, X_0)$ is selfadjoint. Choose $\omega > -\min\sigma(A)$ and set $L := \omega + A$. Denote by (X_α, L_α) for $\alpha \in [0,1]$ the Banach scale generated by fractional powers of L, endowed with corresponding scalar products $(\cdot,\cdot)_\alpha$ and norms $\|\cdot\|_\alpha$. Define $\Gamma\colon X_{1/2} \setminus \{0\} \to \mathbb{R}$ by

$$\Gamma(u) := \frac{\|u\|_{1/2}^2}{\|u\|_0^2} .$$

This function plays a key rôle in deriving the following unique continuation result which we use in Appendix B. It is a variation of [15, Lem. 5.1] (see also [47, III.6] and [34]).

A.16 Lemma. *Suppose that $u \in C([t_0,t_1), X_1) \cap C^1((t_0,t_1), X_0)$, some $t_0 \in \mathbb{R}$ and $t_1 \in (t_0, \infty]$, satisfies $u(t) \neq 0$ for $t \in [t_0,t_1)$. Moreover suppose that there is $h \in L_2((t_0,t_1))$ such that*

$$\|\dot u(t) + Au(t)\|_0 \le h(t)\|u(t)\|_{1/2}$$

for $t \in (t_0,t_1)$. Then for $t \in [t_0,t_1)$ it holds that

$$\Gamma(u(t)) \le \Gamma(u(t_0)) e^{\frac12 \|h\|_{L_2}^2} \tag{A.32}$$

and

$$\|u(t)\|_0 \ge C_1 \|u(t_0)\|_0 e^{-C_2(t-t_0)} \tag{A.33}$$

with constants

$$C_1 := e^{-\frac12 \|h\|_{L_2}^2}$$
$$C_2 := -\omega + \frac32 \Gamma(u(t_0)) e^{\frac12 \|h\|_{L_2}^2} .$$

Proof. First we remark that for $v \in X_1$

$$\Gamma(v) = \frac{(L^{1/2}v, L^{1/2}v)_0}{\|v\|_0^2} = \frac{(Av, v)_0}{\|v\|_0^2} + \omega .$$

Now put $\gamma(t) := \Gamma(u(t))$ for $t \in [t_0,t_1)$ and $f(t) := \dot u(t) + Au(t)$ for $t \in (t_0,t_1)$. To simplify notation we set $|\cdot| := \|\cdot\|_0$ and $(\cdot,\cdot) := (\cdot,\cdot)_0$. The equality

$$\begin{aligned}
&(Au(t+s), u(t+s)) - (Au(t), u(t)) \\
&\qquad = (Au(t+s), u(t+s) - u(t)) + (A(u(t+s) - u(t)), u(t)) \\
&\qquad = (A(u(t+s) + u(t)), u(t+s) - u(t))
\end{aligned}$$

reveals that $t \mapsto (Au(t), u(t))$ is differentiable with

$$\frac{d}{dt}(Au(t), u(t)) = 2(Au(t), \dot u(t))$$

for $t \in (t_0, t_1)$. It follows from Cauchy-Schwarz's inequality that

$$\begin{aligned}
\frac{1}{2}|u|^4\dot{\gamma} &= |u|^2(Au, \dot{u}) - (u, \dot{u})(Au, u) \\
&= |u|^2(Au, f - Au) + (u, Au - f)(Au, u) \\
&= -|u|^2|Au - f/2|^2 + (u, Au - f/2)^2 + \frac{1}{4}|u|^2|f|^2 - \frac{1}{4}(u, f)^2 \\
&\leq \frac{1}{4}|u|^2|f|^2 \\
&\leq \frac{1}{4}|u|^2\|u\|_{1/2}^2 h^2
\end{aligned}$$

and hence

$$\dot{\gamma}(t) \leq \frac{1}{2}h^2(t)\gamma(t)$$

for $t \in (t_0, t_1)$. This proves (A.32).

Now we calculate using (A.32)

$$\begin{aligned}
\frac{d}{dt}\log|u| = \frac{1}{2}\frac{d}{dt}\log|u|^2 = \frac{(u, \dot{u})}{|u|^2} &= \frac{(u, f - Au)}{|u|^2} \\
&= -\gamma + \omega + \frac{(u, f)}{|u|^2} \\
&\geq -\gamma + \omega - h\sqrt{\gamma} \\
&\geq -\frac{3}{2}\gamma + \omega - \frac{1}{2}h^2 \\
&\geq -\frac{3}{2}\gamma(t_0)e^{\frac{1}{2}\|h\|_{L_2}^2} + \omega - \frac{1}{2}h^2
\end{aligned}$$

and (A.33) follows. □

B. The concrete realization

Let $N \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with C^∞-boundary. In all spaces of distributions on Ω we omit the set Ω from the symbol representing the space.

Define the linear boundary value problem $(\mathcal{A}, \mathcal{B})$ by

$$\begin{aligned}
\mathcal{A}u &= -\Delta u \\
\mathcal{B}u &= \gamma_\partial u \ ,
\end{aligned}$$

where γ_∂ is restriction to $\partial\Omega$. Then $(\mathcal{A}, \mathcal{B})$ is normally elliptic [2, Ex. 4.3(e), Rem. 7.3].

For $q > 1$ let $\Sigma_q := \mathbb{Z} + 1/q$ and denote by $H^s_{q,\mathcal{B}}$ for $s \in [-2, 2] \setminus \Sigma_q$ the Bessel potential scale induced by $(\mathcal{A}, \mathcal{B})$ [2, §7]. For $\alpha \in [-1, \infty)$ denote by $(E_{q,\alpha}, A_{q,\alpha})$ the extrapolation-interpolation scale generated by the realization $A_q := A_{q,0} \in \mathcal{L}(H^2_{q,\mathcal{B}}, L_q)$ of $(\mathcal{A}, \mathcal{B})$ in

$E_q := E_{q,0} := L_q$, and the complex interpolation functor $[\cdot,\cdot]_\theta$. By [2, Thm. 7.1] we then have

(B.1) $$E_{q,\alpha} = H^{2\alpha}_{q,\mathcal{B}} \qquad \text{for } 2\alpha \in [-2,2] \setminus \Sigma_q .$$

It is known under these conditions that A_q has bounded imaginary powers, so that the scale $(E_{q,\alpha}, A_{q,\alpha})$ is equivalent to the fractional power scale generated by (E_q, A_q) and possesses the reiteration property. For results about bounded imaginary powers we refer the reader to [36, 44] and [2, Rem. 7.3].

For a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ denote by $\hat{f}$ the superposition operator induced by f, i.e. for $u : \Omega \to \mathbb{R}$ define $\hat{f}(u) : \Omega \to \mathbb{R}$ by $\hat{f}(u)(x) := f(x, u(x))$. It is standard to prove:

B.1 Lemma. *If f satisfies* (F1)*, then $\hat{f} \in C^1(L_r, L_{r/(p-1)})$ uniformly on bounded subsets of L_r, for all $r \geq p-1$. In fact, for $u \in L_r$*

$$\|D\hat{f}(u)\|_{\mathcal{L}(L_r, L_{r/(p-1)})} \leq C(1 + |u|_r^{p-2}) .$$

The differential $D\hat{f}(0)$ is given by $(D\hat{f}(0)u)(x) := f_u(x,0)u(x)$.

In what follows we denote by $\mathcal{F}$ the linear operator that maps a function u to the function given by $x \mapsto f_u(x,0)u(x)$.

We consider problem (P) for $u_0 \in H^1_{q,\mathcal{B}}$.

B.2 Theorem. *Suppose that f satisfies* (F1)*. For every $q \geq 2$ there are $\kappa(q) \in (-1, 0]$ and $\alpha(q) \in [0,1)$ such that setting $X_{q,0} := H^{\kappa(q)}_{q,\mathcal{B}}$, $X_{q,1} := H^{\kappa(q)+2}_{q,\mathcal{B}}$, and $X_{q,\alpha(q)} := [X_{q,0}, X_{q,1}]_{\alpha(q)}$, we have $X_{q,\alpha(q)} = H^1_{q,\mathcal{B}}$, and $\hat{f} \in C^1(X_{q,\alpha(q)}, X_{q,0})$ uniformly on bounded subsets. Moreover, denoting by A the corresponding realization of $(\mathcal{A}, \mathcal{B})$ in $X_{q,0}$, $A \in \mathcal{H}(X_{q,1}, X_{q,0})$.*

For fixed q, the abstract Cauchy problem

(B.2) $$\begin{cases} \dot{u}(t) + Au(t) = \hat{f}(u(t)) & t > 0 \\ u(0) = u_0 & u_0 \in X_{q,\alpha(q)} \end{cases}$$

generates a compact continuous semiflow φ on a domain $\mathcal{D} \subseteq \mathbb{R}^+_0 \times X_{q,\alpha(q)}$ with the properties listed in Theorem A.3*. Let $T_+ : X_{q,\alpha(q)} \to (0,\infty]$ denote the maximal existence time. Then φ has the following additional properties:*

a) *$\varphi : \dot{\mathcal{D}} \to C^1(\overline{\Omega})$ is continuous, and it is continuously differentiable in its second argument.*

b) *For fixed $T \in (0,\infty]$, $V \subseteq T_+^{-1}((T,\infty])$, and $\varepsilon \in [0,T)$ define $M(\varepsilon)$ as in Theorem* A.3g)*. If $M(\varepsilon_1)$ is bounded in $X_{q,\alpha(q)}$ for some $\varepsilon_1 \in (0,T)$, then $M(\varepsilon_2)$ is bounded in $H^2_{r,\mathcal{B}}$ and precompact in $C^1(\overline{\Omega})$ for all $\varepsilon_2 \in (\varepsilon_1, T)$ and $r \geq 2$.*

c) *If $t > 0$, $u, v \in \mathcal{D}_t$ and $u - v \in \mathcal{P}X_{q,\alpha(q)} \setminus \{0\}$, then $\varphi(t,u) - \varphi(t,v)$ lies in $\mathcal{P}_0C^1(\overline{\Omega})$. Similarly, if $t > 0$, $u \in \mathcal{D}_t$, $v \in \mathcal{P}X_{q,\alpha(q)} \setminus \{0\}$, then $D\varphi^t(u)v$ lies in $\mathcal{P}_0C^1(\overline{\Omega})$.*

d) *For every* $t \geq 0$ *and every* $u \in \mathcal{D}_t$, φ^t *and* $D\varphi^t(u)$ *are injective.*

e) *For every* $t \geq 0$ *and every* $u \in \mathcal{D}_t$, $D\varphi^t(u) \in \mathcal{L}(X_{q,\alpha(q)})$ *has dense range.*

f) *If* f *satisfies* (F2), *the spectra and eigenspaces of the operators* $A - D\hat{f}(0)$ *in* $X_{q,0}$ *and* $-\Delta - \mathcal{F}$ *in* L_2 *coincide.*

Proof. We start by exhibiting the scale $X_{q,\gamma}$.

CASE 1: $2 \leq q < N(p-2)/(p-1)$. First observe that we have

$$p < 2^* = \frac{2N}{N-2} \leq \frac{2N}{N-q} \ . \tag{B.3}$$

Also, from $N > 2$ we find $q \geq 2 > N/(N-1)$ and therefore, using (B.3):

$$\frac{qN}{(p-1)(N-q)} > \frac{qN}{N+q} > 1 \ . \tag{B.4}$$

Now set

$$r := \frac{qN}{N-q} \quad \text{and} \quad \theta := N\left(\frac{1}{q} - \frac{p-1}{r}\right)$$

From the assumption on q and (B.3) we find

$$0 > \theta > -1 \ . \tag{B.5}$$

Choose

$$\kappa(q) \in \left[\frac{-1+\theta}{2}, \theta\right] \setminus \Sigma_q \subseteq (-1,0) \setminus \Sigma_q$$

so that

$$\kappa(q) + 1 \geq \frac{\theta+1}{2} > 0 \ . \tag{B.6}$$

Now we have from the definition of r and θ, using (B.3) and (B.4):

$$1 > \frac{1}{q} > \frac{1}{r} > 0 \ , \qquad 1 - \frac{N}{q} \geq 0 - \frac{N}{r}$$

and

$$1 > \frac{p-1}{r} \geq \frac{1}{q} > 0 \ , \qquad 0 - \frac{N(p-1)}{r} = \theta - \frac{N}{q} \geq \kappa(q) - \frac{N}{q} \ .$$

From [2, Eq. (5.9)] it follows that

$$H_q^1 \hookrightarrow L_r \xrightarrow{\hat{f}} L_{r/(p-1)} \hookrightarrow H_q^{\kappa(q)} = H_{q,\mathcal{B}}^{\kappa(q)} \ . \tag{B.7}$$

The last equality is a consequence of [2, Eq. (7.4)]. For $\gamma \in [0,1]$ put $X_{q,\gamma} := E_{q,\gamma+\kappa(q)/2}$ and set

$$\alpha(q) := \frac{1-\kappa(q)}{2} \in (1/2, 1) .$$

Then by (B.1)

$$X_{q,0} = H^{\kappa(q)}_{q,\mathcal{B}}, \quad X_{q,\alpha(q)} = H^1_{q,\mathcal{B}}, \quad X_{q,1} = H^{\kappa(q)+2}_{q,\mathcal{B}} ,$$

and by the characterization of the spaces $E_{q,\gamma}$ given in [2, Sect. 7], together with (B.7) and Lemma B.1, $\hat{f} \in C^1(X_{q,\alpha(q)}, X_{q,0})$ uniformly on bounded sets.

CASE 2: $N(p-2)/(p-1) \le q$. Setting $r := q(p-1)$ and $\kappa(q) := 0$, it follows that

$$1 > \frac{1}{q} > \frac{1}{r} > 0 , \qquad 1 - \frac{N}{q} \ge 0 - \frac{N}{r} .$$

Again from [2, Eq. (5.9)]

$$H^1_q \hookrightarrow L_r \xrightarrow{\hat{f}} L_q .$$

For $\gamma \in [0,1]$ put $X_{q,\gamma} := E_{q,\gamma}$ and set $\alpha(q) := 1/2$. Then by (B.1)

$$X_{q,0} = L_q, \quad X_{q,\alpha(q)} = H^1_{q,\mathcal{B}}, \quad X_{q,1} = H^2_{q,\mathcal{B}}$$

and $\hat{f} \in C^1(X_{q,\alpha(q)}, X_{q,0})$ uniformly on bounded sets.

In any case, by the reiteration property for the scale $(E_{q,\gamma}, A_{q,\gamma})$ we have that $X_{q,\gamma} = [X_{q,0}, X_{q,1}]_\gamma$ for $\gamma \in (0,1)$. Moreover, $A_{q,\kappa(q)/2}$ is the realization (in the sense of [3, p. 7]) of $A_{q,-1}$ in $X_{q,0}$. Consider the abstract initial value problem

(B.8)
$$\begin{cases} \dot{u}(t) + A_{q,\kappa(q)/2} u(t) = \hat{f}(u(t)) & t > 0 \\ u(0) = u_0 & u_0 \in X_{q,\alpha(q)} . \end{cases}$$

By the standard theory outlined in Appendix A, (B.8) generates a compact continuous semiflow on $X_{q,\alpha(q)}$ with the properties listed in Theorem A.3.

For the remaining properties of the semiflow, we need to show how φ regularizes. In order to establish an appropriate bootstrapping argument, define

$$\bar{q}(q) := \begin{cases} \dfrac{2qN}{2N - (\kappa(q)+1)q} & 2 \le q < N(p-2)/(p-1) \\[2ex] \dfrac{2qN}{2N-q} & N(p-2)/(p-1) \le q < 2N \\[2ex] 2q & 2N \le q . \end{cases}$$

We claim that

(B.9)
$$\inf_{q \ge 2} (\bar{q}(q) - q) > 0 .$$

We will show this fact separately on each of the intervals $I_1 := [2, N(p-2)/(p-1))$, $I_2 := [N(p-2)/(p-1), 2N)$, and $I_3 := [2N, \infty)$. For $q \in I_1$ we have in view of (B.6)

$$\bar{q}(q) - q \geq \frac{4qN}{2N + p(N-q)} - q = q\left(\frac{2N - p(N-q)}{2N + p(N-q)}\right) .$$

The last term is continuous, positive and increasing in q on I_1 by (B.3). Hence it is bounded away from 0. If $q \in I_2$ then

$$\bar{q}(q) - q = \frac{q^2}{2N - q}$$

is continuous, positive and increasing in q on I_2, hence bounded away from 0. On I_3 the assertion is obvious. Thus we have proved (B.9).

Now we choose $\beta(q) \in \mathbb{R}$ with

$$2\beta(q) + \kappa(q) \in \left[\frac{3 + \kappa(q)}{2}, 2 + \kappa(q)\right) \setminus \Sigma_q .$$

It follows that

$$\beta(q) \in \left[\frac{1 + \alpha(q)}{2}, 1\right) \subseteq (\alpha(q), 1) .$$

Moreover

$$1 - (2\beta(q) + \kappa(q)) + \frac{N}{q} \leq 1 - \frac{3 + \kappa(q)}{2} + \frac{N}{q} \leq \frac{N}{\bar{q}(q)}$$

holds for $q \geq 2$, so that finally

$$1 > \frac{1}{q} > \frac{1}{\bar{q}(q)} > 0 , \qquad 2\beta(q) + \kappa(q) - \frac{N}{q} \geq 1 - \frac{N}{\bar{q}(q)} .$$

As a consequence of [2, Eq. (5.9)] and (B.1), this yields

$$X_{q,\beta(q)} = H^{2\beta(q)+\kappa(q)}_{q,\mathcal{B}} \hookrightarrow H^1_{\bar{q}(q),\mathcal{B}} = X_{\bar{q}(q),\bar{\alpha}(q)} .$$

Here for convenience we set

$$\bar{\alpha}(q) := \alpha(\bar{q}(q)) .$$

We have the following commuting diagram of natural embeddings:

(B.10)

$$\begin{array}{ccc} & & X_{\bar{q}(q),\bar{\alpha}(q)} \\ & \nearrow & \downarrow \\ X_{q,\beta(q)} & \longrightarrow & X_{q,\alpha(q)} \end{array}$$

Moreover, from uniqueness it is clear that an orbit starting at $u \in X_{q,\alpha(q)}$ coincides with the orbit in $X_{q',\alpha(q')}$ for $q' \in [2, q]$.

a) Now we fix some $q \geq 2$ and let $\mathcal{D}$ denote the domain of φ in $X_{q,\alpha(q)}$. Consider an orbit $\varphi(t, u_0)$ starting at some $u_0 \in X_{q,\alpha(q)}$, with existence interval J. From Theorem A.3e) and (B.10) we know that $u \in C(\dot{J}, X_{q,\beta(q)}) \subseteq C(\dot{J}, X_{\bar{q}(q),\bar{\alpha}(q)})$. Repeating this argument, by (B.9) we see that $u \in C(\dot{J}, X_{r,\beta(r)})$ for all $r \geq 2$.

We claim that $\varphi : \dot{\mathcal{D}} \to X_{r,\beta(r)}$ is continuous, and continuously differentiable in the second argument, for all $r \geq 2$. In view of Theorem A.3d) and of (B.10) it suffices to show this for fixed $r \geq q$ under the condition that $\varphi : \dot{\mathcal{D}} \to X_{r,\alpha(r)}$ has these properties. Therefore, fix $(t_0, u_0) \in \dot{\mathcal{D}}$ and also fix $t_1 \in (0, t_0)$. Let V denote an open neighborhood of $\varphi(t_1, u_0)$ in $X_{r,\alpha(r)}$ such that the restriction of $\varphi^{t_0 - t_1}$ to V is continuously differentiable as a map into $X_{r,\beta(r)}$. This is possible by Theorem A.3f). Let U denote an open neighborhood of (t_1, u_0) in $\dot{\mathcal{D}}$ such that $\varphi(U) \subseteq V$ and such that φ is continuously differentiable in the second argument on U. This is possible since we assume that $\varphi : \dot{\mathcal{D}} \to X_{r,\alpha(r)}$ is continuous, and continuously differentiable in the second argument. Put

$$W := \{ (t, u) \in \mathbb{R}_0^+ \times X_{q,\alpha(q)} \mid (t - t_0 + t_1, u) \in U \} .$$

Then W is an open neighborhood of (t_0, u_0) in $\dot{\mathcal{D}}$ and

$$\varphi(t, u) = \varphi(t_0 - t_1, \varphi(t - t_0 + t_1, u))$$

for all $(t, u) \in W$. From this it is clear that $\varphi : W \to X_{r,\beta(r)}$ is continuous, and continuously differentiable in the second argument. Since $(t_0, u_0) \in \dot{\mathcal{D}}$ was arbitrary, this proves the claim. Choosing r large enough such that $X_{r,\beta(r)} \subseteq C^1(\overline{\Omega})$ we have proved a).

b) The statement on boundedness and compactness follows from Theorem A.3g) by the bootstrapping procedure outlined above, and by the compactness of the embeddings (B.10).

c) The comparison principle is proved in a standard way, see e.g. [18]. Note that due to our weak regularity assumptions on the coefficients of (P) some approximation arguments have to be used in order to apply the results from [18].

d) To show backward uniqueness, assume that for some $t_0 > 0$ and $u_0, v_0 \in \mathcal{D}_{t_0}$ with $u_0 \neq v_0$ we have $\varphi(t_0, u_0) = \varphi(t_0, v_0)$. Let u, v denote the orbits starting in u_0, v_0. Going forward in time a small amount we may assume that

$$u, v \in C([0, t_0], C^1(\Omega)) \cap C([0, t_0], H^2_{2,\mathcal{B}}) \cap C^1([0, t_0], L_2) .$$

We may also assume that t_0 is the first time such that $u(t_0) = v(t_0)$. Since u, v are bounded in $C(\overline{\Omega})$, there is $M \geq 0$ such that

$$g(t) := \hat{f}(u(t)) - \hat{f}(v(t))$$

satisfies

$$\|g(t)\|_{L_2} \leq M \|u(t) - v(t)\|_{L_2}$$

for $t \in [0, t_0]$. This follows from (F1). Setting $w := u - v$, w is a solution of

$$\dot{w}(t) + Aw(t) = g(t) ,$$

where A is the realization of $-\Delta$ in L_2. Now Lemma A.16 yields that $w(t_0) \neq 0$, a contradiction. The proof of injectivity of $D\varphi^t(u)$ is similar. This proves d).

Property e), i.e. that $D\varphi^t(u)$ has dense range in this setting, is proved in [1]. For $q > N$ and under stronger assumptions on f this has also been considered in [25, Ex., p. 209], although the proof seems to be incomplete.

f) For $q \geq 2$ define $B_{q,0} := A_{q,0} - \mathcal{F}$. Note that $\mathcal{F} \in \mathcal{L}(L_q)$, since by (F1) $f_u(\cdot, 0) \in L_\infty$. As we have shown in Appendix A.3.2, $B_{q,0} \in \mathcal{H}(E_{q,1}, E_{q,0})$. From the definition of the adjoint of a densely defined closed operator it easily follows that $\mathrm{dom}(B'_{q,0}) = \mathrm{dom}(A'_{q,0})$, considered as operators in $\mathcal{C}(E_{q,0})$. Therefore, by [3, Thm. V.2.1.3], $B_{q,0}$ is closable in $E_{q,-1}$. We denote its closure by $B_{q,-1}$. Moreover, $B_{q,-1} \in \mathcal{H}(E_{q,0}, E_{q,-1})$. We can define for $\alpha \in [-1, 0]$ the realization $B_{q,\alpha}$ of $B_{q,-1}$ in $E_{q,\alpha}$. Then $\sigma(B_{q,\alpha})$ is independent of $\alpha \in [-1, 0]$. Again $B_{q,\alpha} \in \mathcal{H}(E_{q,\alpha+1}, E_{q,\alpha})$.

Recall that if X is a Banach space, $A \in \mathcal{C}(X)$, $\rho(A) \neq \emptyset$, $D(A)$ is $\mathrm{dom}(A)$ equipped with the graph norm, then every dense subset of $D(A)$ is a core for A (see [29, III.6.1, Problem 6.3]). Since the scale $(E_{q,\alpha})_\alpha$ is densely embedded, $E_{q,1} = H^2_{q,\mathcal{B}}$ is a core for $A_{q,\alpha}$ and $B_{q,\alpha}$.

We have defined $X_{q,0} = E_{q,\kappa(q)/2}$, $X_{q,1} = E_{q,1+\kappa(q)/2}$, and $X_{q,\alpha(q)} = E_{q,1/2}$, so that $\hat{f} \in C^1(X_{q,\alpha(q)}, X_{q,0})$. As before,

$$\widetilde{B}_q := A_{q,\kappa(q)/2} - D\hat{f}(0) \in \mathcal{H}(E_{q,1+\kappa(q)/2}, E_{q,\kappa(q)/2}) ,$$

and $\widetilde{B}_q$ coincides with $B_{q,0}$ on $E_{q,1}$ by Lemma B.1. By the same reasoning as above, $E_{q,1}$ is a core for $\widetilde{B}_q$. Hence $\widetilde{B}_q = B_{q,\kappa(q)/2}$. It is easy to see that the eigenspaces of $B_{q,\alpha}$ are independent of $\alpha \in [-1, 0]$. This follows from the properties of Banach scales.

To prove that the spectral properties are independent of $q \geq 2$, recall that we have embeddings $E_{q,1} \hookrightarrow E_{2,1}$. Thus all eigenvectors of $B_{q,0}$ are also eigenvectors of $B_{2,0}$. In view of the bootstrapping procedure outlined above, and of the independence of α, every eigenvector of $B_{2,0}$ is also an eigenvector of $B_{q,0}$ for $q \geq 2$. Together these observations prove f). $\square$

One can extract some more information from the comparison principle regarding the invariance of certain cones under the semiflow. Recall the definition of $\mathcal{S}^\pm$ and $\mathcal{S}^\pm_{\mathrm{reg}}$ given in Section 2.2.

B.3 Lemma. *Let f satisfy* (F1). *Then for $u \in \mathcal{S}^+$ the set $u - \mathcal{P}H^1_{2,\mathcal{B}}$ is positive invariant with respect to φ, and for $u \in \mathcal{S}^-$ the set $u + \mathcal{P}H^1_{2,\mathcal{B}}$ is positive invariant.*

Proof. If $u \in \mathcal{S}^-_{\mathrm{reg}}$ and $v \in C^2_0(\overline{\Omega})$ satisfies $v \geq u$, the map $(t, x) \mapsto u(x)$ is a subsolution for the parabolic problem (P). From the comparison principle we obtain that $\varphi(t, v) \geq u$ for all $t \in J(v)$. For the general case, we consider $u \in \mathcal{S}^-$, and suppose that $v \in H^1_{2,\mathcal{B}}$ satisfies $v \geq u$. Choose sequences $(u_n) \subseteq \mathcal{S}^-_{\mathrm{reg}}$ and $(w_n) \subseteq C^2_0(\overline{\Omega})$ with $w_n \geq 0$ such that $u_n \to u$ and $w_n \to v - u$ in $H^1_{2,\mathcal{B}}$. Set $v_n := u_n + w_n$ so that $v_n \geq u_n$ for all n. The continuity of φ and the invariance in the regular case proved above then yields

$$\varphi(t, v) - u = \lim_{n\to\infty} (\varphi(t, v_n) - u_n) \geq 0$$

for all $t \in J(v)$. The proof for supersolutions proceeds analogously. $\square$

Let f satisfy (F1) and (F2), and consider the semiflow φ given by Theorem B.2. Suppose that u_0, u_1, u_2 are orbits of φ existing for all $t \geq 0$ such that $u_i(t) \to 0$ in $X_{2,\alpha(2)} = H^1_{2,\mathcal{B}}$ as $t \to \infty$, $i = 0, 1, 2$. Due to Theorem A.3e) and Theorem B.2a), $u_i(t) \to 0$ in $C^1(\overline{\Omega})$ and in $X_{q,\gamma}$ for all $q \geq 2$ and $\gamma \in [0, 1)$. Set

(i) $v(t) := D\varphi^t(u_0(0))v_0$ for some $v_0 \in X_{2,\alpha(2)}$ or

(ii) $v(t) := u_1(t) - u_2(t)$.

Moreover, suppose that $v(0) \neq 0$. By Theorem B.2d), $v(t) \neq 0$ for $t \geq 0$.

In our setting the linearization $-\Delta - \mathcal{F}$ has compact resolvent so that by Corollary A.11b) $\lim_{t\to\infty} \|u_i(t)\|^{1/t}_{X_{2,\alpha(2)}}$ and $\lim_{t\to\infty} \|v(t)\|^{1/t}_{X_{q,\gamma}}$ exist for all $q \geq 2$ and $\gamma \in [0, 1)$. Moreover, $\lim_{t\to\infty} \|u_i(t)\|^{1/t}_{X_{2,\alpha(2)}} \leq 1$ for $i = 1, 2, 3$.

B.4 Lemma. *Set* $a := \lim_{t\to\infty} \|v(t)\|^{1/t}_{X_{2,\alpha(2)}} \in \mathbb{R}^+_0$.

a) *If* $v(t)$ *is as above, then for every* $q \geq 2$ *and every* $\gamma \in [0, 1)$

$$\lim_{t\to\infty} \|v(t)\|^{1/t}_{C^1(\overline{\Omega})} = \lim_{t\to\infty} \|v(t)\|^{1/t}_{X_{q,\gamma}} = a\ .$$

b) *Suppose that in addition* (F4) *holds. If* $\lim_{t\to\infty} \|u_0(t)\|^{1/t}_{X_{2,\alpha(2)}} < 1$ *in case* (i)*, respectively* $\lim_{t\to\infty} \|u_i(t)\|^{1/t}_{X_{2,\alpha(2)}} < 1$ *for* $i = 1$ *or* $i = 2$ *in case* (ii)*, then* $a > 0$.

Proof. a) From Corollary A.11b) we know that

$$\lim_{t\to\infty} \|v(t)\|_{X_{2,\gamma}} = a$$

for $\gamma \in [0, 1)$. In view of (B.10) there are constants $C_1, C_2 \geq 1$ such that

$$\|\cdot\|_{X_{2,\alpha(2)}} \leq C_1 \|\cdot\|_{X_{\bar{q}(2),\bar{\alpha}(2)}} \qquad \text{and} \qquad \|\cdot\|_{X_{\bar{q}(2),\bar{\alpha}(2)}} \leq C_2 \|\cdot\|_{X_{2,\beta(2)}}\ .$$

Thus

$$\|v(t)\|^{1/t}_{X_{\bar{q}(2),\bar{\alpha}(2)}} \geq C_1^{-1/t} \|v(t)\|^{1/t}_{X_{2,\alpha(2)}} \to a$$
$$\|v(t)\|^{1/t}_{X_{\bar{q}(2),\bar{\alpha}(2)}} \leq C_2^{1/t} \|v(t)\|^{1/t}_{X_{2,\beta(2)}} \to a$$

as $t \to \infty$. Hence

$$\lim_{t\to\infty} \|v(t)\|^{1/t}_{X_{\bar{q}(2),\bar{\alpha}(2)}} = a\ .$$

Again Corollary A.11b) yields

$$\lim_{t\to\infty} \|v(t)\|^{1/t}_{X_{\bar{q}(2),\gamma}} = a$$

for all $\gamma \in [0, 1)$. Repeating this argument we obtain

$$\lim_{t\to\infty} \|v(t)\|^{1/t}_{X_{q,\gamma}} = a$$

for all $q \geq 2$ and $\gamma \in [0, 1)$.

Observe that

$$\begin{array}{ccc} & & C^1(\overline{\Omega}) \\ & \nearrow & \downarrow \\ X_{q,\beta(q)} & \longrightarrow & X_{q,\alpha(q)} \end{array}$$

commutes if q is large enough. By the same argument as above we obtain

$$\lim_{t\to\infty} \|v(t)\|_{C^1(\overline{\Omega})}^{1/t} = a \ .$$

b) Note that applying a) to $u_i(t) - 0$ we obtain

$$\lim_{t\to\infty} \|u_i(t)\|_{C^1(\overline{\Omega})}^{1/t} = \lim_{t\to\infty} \|u_i(t)\|_{X_{2,\alpha(2)}}^{1/t} \ .$$

In particular, if $\lim_{t\to\infty} \|u_i(t)\|_{X_{2,\alpha(2)}}^{1/t} < 1$ ($i = 0, 1, 2$), then

$$\|u_i(t)\|_{C(\overline{\Omega})} \text{ decays exponentially fast as } t \to \infty. \tag{B.11}$$

First we prove the claim in the case (i). Recall that v solves the equation

$$\dot{v}(t) + (-\Delta - f_u(\cdot, 0))v(t) = (f_u(\cdot, u_0(t)(\cdot)) - f_u(\cdot, 0))v(t) \tag{B.12}$$

for $t > 0$. Going forward in time a small amount we may assume that (B.12) holds for $t \geq 0$. By (F4) and (B.11) also $h(t) := \|f_u(\cdot, u_0(t)(\cdot)) - f_u(\cdot, 0)\|_{L_\infty}$ decays exponentially fast as $t \to \infty$, so that $h \in L_2((0, \infty))$ and

$$\|\dot{v}(t) + (-\Delta - \mathcal{F})v(t)\|_{L_2} \leq h(t)\|v(t)\|_{L_2}$$

for $t \geq 0$. Hence $a > 0$ by Lemma A.16.

To prove b) in case (ii) suppose first that $\lim_{t\to\infty} \|u_1(t)\|_{X_{2,\alpha(2)}}^{1/t} = 1$. Then by our assumptions

$$\frac{\|u_2(t)\|_{X_{2,\alpha(2)}}}{\|u_1(t)\|_{X_{2,\alpha(2)}}} \to 0$$

as $t \to \infty$, and the claim follows easily. The same proof applies to the case $\lim_{t\to\infty} \|u_2(t)\|_{X_{2,\alpha(2)}}^{1/t} = 1$.

Now suppose that $\lim_{t\to\infty} \|u_1(t)\|_{X_{2,\alpha(2)}}^{1/t} < 1$ and $\lim_{t\to\infty} \|u_2(t)\|_{X_{2,\alpha(2)}}^{1/t} < 1$. Set

$$g(t, x) := \int_0^1 f_u(x, su_1(t)(x) + (1-s)u_2(t)(x))\, ds - f_u(x, 0) \ .$$

Then v satisfies the equation

$$\dot{v}(t) + (-\Delta - f_u(\cdot, 0))v(t) = g(t, \cdot)v(t) \tag{B.13}$$

for $t > 0$, and again we may assume that (B.13) is even satisfied for $t \geq 0$. As above, from (F4), (B.11) and Lemma A.16 it follows that $a > 0$. □

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