Ground States for Irregular and Indefinite Superlinear Schrödinger Equations*

Nils Ackermann Julián Chagoya

We consider the existence of a ground state for the subcritical stationary semilinear Schrödinger equation $-\Delta u + u = a(x)|u|^{p-2}u$ in H^1, where $a \in L^\infty(\mathbb{R}^N)$ may change sign. Our focus is on the case where loss of compactness occurs at the ground state energy. By providing a new variant of the Splitting Lemma we do not need to assume the existence of a limit problem at infinity, be it in the form of a pointwise limit for a as $|x| \to \infty$ or of asymptotic periodicity. That is, our problem may be *irregular* at infinity. In addition, we allow a to change sign near infinity, a case that has never been treated before.

Keywords: Stationary Schrödinger Equation, Ground State, Indefinite Superlinear, Subcritical, No Limit Problem

1 Introduction

We are concerned with the subcritical stationary semilinear Schrödinger equation

$$-\Delta u + u = a(x)|u|^{p-2}u, \qquad u \in H^1(\mathbb{R}^N), \tag{1.1}$$

where $H^1 := H^1(\mathbb{R}^N)$ is the usual Sobolev space and $a \in L^\infty(\mathbb{R}^N)$. Here and in what follows function spaces are over $\mathbb{R}^N$ unless otherwise noted. Suppose throughout that $2 < p < 2^*$, where $2^* := 2N/(N-2)$ if $N \geq 3$, $2^* := \infty$ if $N = 1$ or 2, is the critical Sobolev exponent.

Solutions to the more general problem

$$-\Delta u + V(x)u = a(x)|u|^{p-2}u, \qquad u \in H^1(\mathbb{R}^N), \tag{1.2}$$

*This research was partially supported by CONACYT grant 237661 and UNAM-DGAPA-PAPIIT grant IN104315 (Mexico)

give rise to certain solitary waves of the corresponding time dependent Schrödinger or Klein-Gordon equations, and have therefore received much attention in the literature. Under appropriate conditions on V and a, weak solutions of (1.2) are in correspondence with the critical points of the variational functional (the "energy") $J\colon H^1 \to \mathbb{R}$ defined by

$$J(u) := \frac{1}{2}\int(|\nabla u|^2 + V|u|^2) - \frac{1}{p}\int a|u|^p.$$

A *ground state* of (1.2) is a minimum of J on the Nehari manifold

$$\{u \in H^1\backslash\{0\} \mid \mathrm{D}J(u)u = 0\},$$

which is a nontrivial critical point of J under suitable conditions on V and a. The problem of existence of ground states for (1.2) is of particular interest since they potentially yield orbitally stable standing wave solutions to the Schrödinger Equation [32, 49]. The main obstacle to prove existence of solutions for (1.2) is the inherent lack of compactness, that is, the failure of the Palais-Smale or Cerami conditions for J due to the noncompact embedding $H^1 \hookrightarrow L^p$ for $p \in (2, 2^*)$.

If V and a are constant then existence of ground states of (1.1) was analyzed in the seminal work of Berestycki and Lions [7], see also the references therein. For the nonautonomous equation, existence of ground states has been considered under various hypotheses to overcome the lack of compactness.

If V and a are radially symmetric then compactness is restored in the radially symmetric subspace H^1_{r} of H^1 [5, 25]. Depending on other properties of V and a a ground state in H^1_{r} may or may not be a ground state in H^1. If, roughly speaking, $\lim_{|x|\to\infty} V(x) = \infty$, $\limsup_{|x|\to\infty} a(x) \le 0$ or $a^+ \in L^q$ for a suitable $q > 0$ then compactness is restored in H^1 or in an appropriately weighted space [2, 20, 21, 23, 27–29, 31, 33, 39, 44, 47, 48, 52]. And, last but not least, replacing the right hand side of (1.2) by $f(x, u)$, where f is asymptotically linear in u, a nonresonance condition ensures compactness [40, 58].

Apart from these cases, most results impose the existence of a limit problem at infinity and employ concentration compactness arguments. This can be achieved by assuming the existence of pointwise limits of V and a as $|x| \to \infty$, see, e.g., [3, 4, 10–12, 14, 15, 17, 22, 25, 29, 30, 35, 37, 41, 54–56] or, more generally, [16, 18, 34]. Another variant of this approach is to assume (asymptotic) periodicity of V and a in the coordinates of the x variable, see, e.g., [24, 26, 33, 35, 36, 43, 50, 51, 53, 57].

The only existence result for (1.2) in the setting without compactness we are aware of that does not impose a limit on V and a as $|x| \to \infty$ is [13]. Here the existence of a ground state is shown for $a \equiv 1$ and $\operatorname{ess\,inf} V > 0$, assuming that V takes values below $\liminf_{|x|\to\infty} V$ on a large enough ball (Theorem 1.2 *cit. loc.*). The condition is not explicit though and cannot be checked directly. Nevertheless, under explicit conditions on V the authors prove in Theorem 1.3 *cit. loc.* the existence of a solution, which is not a ground state.

Another important aspect is the sign of the functions V and a. We say that (1.2) is *linearly indefinite* if V changes sign, and *superlinearly indefinite* if a changes sign. There are many existence results for the superlinearly indefinite problem, linearly indefinite or not, see [18, 20–23, 27–29, 31, 34, 39, 47, 52]. In all of these results a is not allowed to change sign near infinity, in some sense. One often used assumption is that either $\limsup_{|x|\to\infty} a(x) \leq 0$ or $\liminf_{|x|\to\infty} a(x) \geq 0$. More generally, a number of authors assume that either a^+ or a^- belongs to some L^q-space for suitable q, or variants of this type of hypothesis.

Our aim in this article is twofold: first we want to remove the assumption of pointwise limits at infinity or of asymptotic periodicity for the functions V and a in the scenario without compactness. This is what we allude to with the term *irregular* in the title. And second, we are interested in lifting the requirement that a does not change sign at infinity in the indefinite superlinear problem. To highlight the second goal we restrict ourselves and assume $V \equiv 1$ for the rest of the text.

Before presenting our results we need to introduce some more notation. If $a \in L^\infty$ then we define the variational functional

$$J_a(u) := \frac{1}{2}\int (|\nabla u|^2 + |u|^2) - \frac{1}{p}\int a|u|^p$$

corresponding to (1.1). For $y \in \mathbb{R}^N$ we define the translation operator τ_y on spaces of functions on $\mathbb{R}^N$ by

$$(\tau_y f)(x) := f(x-y).$$

Denote for $q \in [1,\infty]$ by $|\cdot|_q$ the norm in L^q. In a normed space X denote by $B_r(x) := B_r(x;X)$ and $\overline{B}_r(x) := \overline{B}_r(x;X)$ the open and closed balls with radius $r>0$ and center $x \in X$. If $x=0$ we set $B_rX := B_r(0;X)$ and $\overline{B}_rX := \overline{B}_r(0;X)$. Fixing $a \in L^\infty$ define the set

$$\mathcal{P} := \overline{B}_{|a|_\infty} L^\infty.$$

We endow $\mathcal{P}$ with the weak*-Topology, identifying L^∞ with the dual space of L^1, and obtain a compact metrizable space (cf. [46, Theorems 3.15 and 3.16]). Consider the subset $\mathcal{A}$ of translates of a,

$$\mathcal{A} := \left\{ \tau_y a \;\middle|\; y \in \mathbb{R}^N \right\} \subseteq \mathcal{P}. \tag{1.3}$$

In dealing with $\mathcal{A}$ we will always use the topology induced by $\mathcal{P}$. Define

$$\mathcal{B} := \overline{\mathcal{A}} \backslash \mathcal{A} \subseteq \mathcal{P} \tag{1.4}$$

(in general this is not the topological boundary of $\mathcal{A}$ in $\mathcal{P}$). We now set

$$\bar{a} := \sup_{u\in\mathcal{B}} (\operatorname{ess\,sup} u). \tag{1.5}$$

By convention, $\bar{a} := -\infty$ if $\mathcal{B} = \varnothing$. It follows from Lemma 2.1 below that $\bar{a} \leq \operatorname{ess\,sup} a$, and the inequality may be strict (if a is constant, $\bar{a} = -\infty$). Denoting by $a^\pm := \max\{0, \pm a\}$ the positive and negative parts of a we introduce the following condition on the function a:

(A) $a^+ \neq 0$ as an element of L^∞ and either (i) $\bar{a} \leq 0$ or (ii) $\bar{a} \leq a$ or (iii) there exist $\gamma \in (0,2)$, sequences $(z_n) \subseteq \mathbb{R}^N$ and $(R_n) \subseteq \mathbb{R}$, and a non-negative function $\kappa \in L^1_{\mathrm{loc}} \backslash \{0\}$ such that $R_n \to \infty$, $\tau_{-z_n} a \overset{\mathrm{w}^*}{\to} \bar{a}$, and $a(x) \geq \bar{a} + \kappa(x - z_n)\mathrm{e}^{-\gamma R_n}$ for all n and $x \in B_{R_n}(z_n)$.

Our main result is the following

Theorem 1.1. *If* $p \in (2, 2^*)$, $a \in L^\infty$, *and* (A) *is satisfied then* J_a *has a positive ground state.*

We allow the situation where $\tau_{x_n} a$ has no pointwise convergent subsequence for any sequence $(x_n) \subseteq \mathbb{R}^N$ such that $|x_n| \to \infty$. Instead, we use the weak*-compactness to pass to a limit problem for each such sequence, by using concentration compactness in the form of a new variant of the Splitting Lemma, which was introduced originally by Benci and Cerami [6]: suppose that (u_n) is a Palais-Smale sequence for J_a at the ground state energy. In an iterative procedure, passing to subsequences implicitly, one splits off parts of this sequence that remain concentrated. Usually one proves in each step that the resulting sequence (v_n) is still Palais-Smale. If we only work with weak*-compactness of sequences $\tau_{x_n} a$ this is no longer true in general. We show instead that a weaker condition is satisfied, that is, that $\nabla J_a(v_n)$ vanishes in H^1, see Definitions 3.1 and 3.2 below. This yields a Splitting Lemma with a slightly weaker statement than usual, which is still sufficient to show the existence of a ground state. The only reference we know where weak*-compactness was used in a similar way is [40]. There the authors show that compactness is restored in a nonresonant asymptotically linear problem, so concentration compactness is not used.

In what follows we will present some tools to easily generate nontrivial examples of functions a that satisfy (A). In order to do that, denote for $a \in L^\infty$

$$\hat{a} := \lim_{R\to\infty} \left(\operatorname{ess\,sup} a|_{\mathbb{R}^N \backslash B_R} \right).$$

In general, $\hat{a} \neq \bar{a}$, as can be seen when a is constant.

Proposition 1.2. *If* (A) *is satisfied with* $\bar{a}$ *replaced by* $\hat{a}$ *then* (*the original condition*) (A) *is satisfied.*

One immediate consequence is that the well known cases $\hat{a} \leq 0$ and $\hat{a} \leq a$ are covered by Theorem 1.1. To illustrate, suppose that $a \in C_{\mathrm{b}}(\mathbb{R}^N)$ (the bounded continuous functions on $\mathbb{R}^N$) satisfies $a^+ \neq 0$. If $\hat{a} = \limsup_{|x|\to\infty} a(x) \leq 0$ then the existence of a ground state was shown in [33]. If $\hat{a} = \limsup_{|x|\to\infty} a(x) \leq a$ then $\hat{a} = \lim_{|x|\to\infty} a(x)$ exists and $\hat{a} \leq a$. Either $a \equiv \hat{a} > 0$ and the radial ground state in H^1_{r} is also a ground state in H^1, since every ground state in H^1 is radially symmetric, or $a \gneq \hat{a}$ and a ground state exists by standard concentration compactness arguments, see [37, 38]. Proposition 1.5 below shows that even for this simple setup our main result gives an improvement since it allows to extend the problem to higher dimensions by constancy.

To illustrate Proposition 1.2 further we provide some nontrivial new examples. The role of the function κ in condition (A) is highlighted by a general hypothesis about the behavior of a in a cone:

Example 1.3. Suppose that $a \in L^\infty$ satisfies $\hat{a} > 0$ and has the following properties: there exist $x_0 \in \mathbb{R}^N$, $|x_0| = 1$, $r_0 \in (0,1)$, $\gamma \in (0, 2r_0)$, and $R > 0$ such that, setting

$$U := \{tx \mid x \in B_{r_0}(x_0),\ t > 0\},$$

it holds true that

$$\lim_{\substack{|x|\to\infty \\ x \in U}} a(x) = \hat{a} \tag{1.6}$$

and

$$\forall x \in U \backslash \overline{B}_R \colon a(x) \geq \hat{a} + \mathrm{e}^{-\gamma|x|}. \tag{1.7}$$

Then a satisfies (A).

Note that outside of the cone U the behavior of a is only restricted by the assumption $\operatorname{ess\,sup}(a|_{\mathbb{R}^N \backslash U}) \leq \hat{a}$.

Proof of Example 1.3. Clearly $a^+ \not\equiv 0$. Put

$$z_n := \left(\frac{R}{1 - r_0} + n - 1\right) x_0 \qquad \text{and} \qquad R_n := r_0 |z_n|.$$

It is easy to see that then

$$B_{R_n}(z_n) \subseteq U \backslash \overline{B}_{R + (1-r_0)(n-1)} \subseteq U \backslash \overline{B}_R. \tag{1.8}$$

Hence we obtain from (1.7) for all $x \in B_{R_n}(z_n)$

$$a(x) \geq \hat{a} + \mathrm{e}^{-\gamma|x|} \geq \hat{a} + \mathrm{e}^{-\gamma|x - z_n|}\mathrm{e}^{-\gamma|z_n|} = \hat{a} + \kappa(x - z_n)\mathrm{e}^{-\frac{\gamma}{r_0}R_n},$$

where we have set $\kappa(x) := \mathrm{e}^{-\gamma|x|}$ and where $\frac{\gamma}{r_0} \in (0, 2)$. Moreover, (1.6) and (1.8) imply that $(\tau_{-z_n} a)(x) \to \hat{a}$ for all $x \in \mathbb{R}^N$. By Lebesgue's Dominated Convergence Theorem, $\tau_{-z_n} a \overset{\mathrm{w}^*}{\to} \hat{a}$. □

In the next example there is no direction in which a converges pointwise as $|x| \to \infty$:

Example 1.4. Define $Z_0 := B_1$ and $Z_i := B_{2^i} \backslash B_{2^{i-1}}$, for $i \in \mathbb{N}$. For any $\gamma \in (0, 2)$ define $a \in L^\infty$ by

$$a(x) := \begin{cases} -1 & x \in Z_i,\ i \text{ odd}, \\ 1 + \mathrm{e}^{-\gamma 2^{i-2}} & x \in Z_i,\ i \text{ even}. \end{cases}$$

Then a satisfies (A).

Proof. For $n \in \mathbb{N}$ take $z_n := (3 \cdot 2^{2n-2}, 0, \dots, 0)$ and $R_n := 2^{2n-2}$. Then $B_{R_n}(z_n) \subseteq Z_{2n}$ and hence
$$\forall x \in B_{R_n}(z_n)\colon a(x) = 1 + \mathrm{e}^{-\gamma 2^{2n-2}} = 1 + \mathrm{e}^{-\gamma R_n}. \qquad \square$$

In the last example the radial symmetry would allow to work in the space H^1_{r} of radially symmetric functions, where we have compactness. Nevertheless, the radial ground state is not necessarily a ground state, so our Theorem 1.1 gives a better result. Of course, one could combine Examples 1.3 and 1.4 into an example where a is not radially symmetric and where there is no direction in which a converges pointwise as $|x| \to \infty$. Moreover, one could replace the concentric circles separating different regions of definition of a in Example 1.4 by other geometric objects, for example by logarithmic spirals.

We now prove another tool to generate examples by extension to higher dimensions:

Proposition 1.5. *Suppose that $k, \ell \in \mathbb{N}$ and that* (A) *is satisfied for $a \in L^\infty(\mathbb{R}^k)$. Define $a'\colon \mathbb{R}^k \times \mathbb{R}^\ell \to \mathbb{R}$ by $a'(x, y) := a(x)$. Then* (A) *is satisfied for $a' \in L^\infty(\mathbb{R}^k \times \mathbb{R}^\ell)$.*

To mention a special case, suppose that $a \in L^\infty(\mathbb{R})$ has limits $a_{\pm\infty} := \lim_{x\to\pm\infty} a(x)$, that $a_{+\infty} > \max\{a_{-\infty}, 0\}$, and that there are $\gamma \in (0, 2)$ and $R > 0$ such that
$$\forall x > R\colon a(x) \geq a_{+\infty} + \mathrm{e}^{-\gamma x}.$$

Choosing $r_0 \in (\gamma/2, 1)$ we obtain from Example 1.3 that a satisfies (A). Hence by Proposition 1.5 also the function $a' \in L^\infty(\mathbb{R}^2)$ defined by $a'(x_1, x_2) := a(x_1)$ satisfies (A), and the problem
$$-\Delta u + u = a'(x)|u|^{p-2}u, \qquad u \in H^1(\mathbb{R}^2) \tag{1.9}$$
has a ground state. In contrast, if $a_{-\infty} > 0$ then by [26, Theorem 3] problem (1.9) does not possess a ground state for the function
$$a'(x_1, x_2) := \begin{cases} a_{-\infty}, & x_1 < 0, \\ a_{+\infty}, & x_1 > 0. \end{cases}$$

With respect to the existence of a ground state our problem is analogous to the case where $a \equiv 1$ and V depends on x. For problem (1.1) one expects to find a ground state roughly when a is strictly larger than $\limsup_{|x|\to\infty} a(x)$ on a large enough set. In the other case one expects to find a ground state roughly when V is strictly smaller than $\liminf_{|x|\to\infty} V(x)$ on a large enough set. In a forthcoming paper we are dealing with the latter case in a similar manner as here, and we also treat the case where both of V and a are not constant.

The article is structured as follows: In Section 2 we explain how the Nehari manifold can be used in indefinite superlinear problems. In Section 3 we prove our new variant of the Splitting Lemma. Section 4 is devoted to the proof of Theorem 1.1, and in Section 5 we prove Propositions 1.2 and 1.5.

2 The Nehari Manifold

We assume in this section that $a \in L^\infty$ and $a^+ \neq 0$. The following facts about the Nehari manifold are known but maybe not as commonly as in the case of $a > 0$. They could also be taken from the references [9, 42].

Denote by $|E|$ the Lebesgue measure of a measurable subset E of $\mathbb{R}^N$. In any normed space X denote by $S_1X := \{x \in X \mid \|x\| = 1\}$ the sphere of radius 1 with center 0. The following will be useful in various places:

Lemma 2.1. *For any $u \in L^\infty$ it holds true that*

$$\operatorname{ess\,sup} u = \sup\Big\{\int_{\mathbb{R}^N} u\varphi \;\Big|\; \varphi \in S_1L^1,\ \varphi \geq 0\Big\}. \tag{2.1}$$

Proof. Denote by c the right hand side of (2.1). If $\varphi \in S_1L^1$ and $\varphi \geq 0$ then

$$\int_{\mathbb{R}^N} u\varphi \leq \operatorname{ess\,sup} u \int_{\mathbb{R}^N} \varphi = \operatorname{ess\,sup} u.$$

Therefore, $c \leq \operatorname{ess\,sup} u$. To show the inverse inequality, first fix some $R > 0$ and $x \in \mathbb{R}^N$. For any Lebesgue-measurable subset $E \subseteq B_R(x)$ we find

$$\frac{1}{|E|}\int_E u = \int_{\mathbb{R}^N} u\frac{\chi_E}{|E|} \leq c,$$

where χ_E denotes the characteristic function of E. From [45, Theorem 1.40] it follows that $u(x) \leq c$ a.e. in $B_R(x)$. Covering $\mathbb{R}^N$ with countably many balls $B_R(x)$ we obtain $u(x) \leq c$ a.e. in $\mathbb{R}^N$, i.e., $\operatorname{ess\,sup} u \leq c$. □

From $a^+ \neq 0$ we obtain $\operatorname{ess\,sup} a > 0$, and Lemma 2.1 yields $\varphi \in S_1L^1$ such that $\varphi \geq 0$ and $\int a\varphi > 0$. Approximating φ suitably in L^1 we obtain that

$$\exists \psi \in C_c^\infty : \psi \geq 0 \qquad \text{and} \qquad \int a\psi > 0. \tag{2.2}$$

We define the Nehari set of J_a by

$$\mathcal{N}_a := \{u \in H^1\backslash\{0\} \mid \mathrm{D}J_a(u)u = 0\}$$

and set $c_a := \inf_{\mathcal{N}_a} J_a$. By convention, $c_a := \infty$ if $\mathcal{N}_a = \varnothing$. Denote by

$$\mathcal{C}_p := \sup_{u \in H^1\backslash\{0\}} \frac{|u|_p}{\|u\|}$$

the operator norm of the continuous embedding $H^1 \hookrightarrow L^p$. Here

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)$$

defines the standard norm on H^1. Set

$$\mathcal{U}_a^+ := \left\{ u \in H^1 \;\middle|\; \int_{\mathbb{R}^N} a|u|^p > 0 \right\}$$

and

$$\mathcal{S}_a^+ := \mathcal{U}_a^+ \cap S_1 H^1.$$

By (2.2) $\mathcal{U}_a^+$ is nonempty. Moreover, $\mathcal{U}_a^+$ is open in H^1 and $\mathcal{S}_a^+$ is open in $S_1 H^1$. For any $u \in H^1 \backslash \{0\}$ we define the *fibering map* $h_{a,u}\colon [0,\infty) \to \mathbb{R}$ by $h_{a,u}(t) := J_a(tu)$. It is then clear that $h_{a,u}$ has a positive critical point if and only if $u \in \mathcal{U}_a^+$. If this is the case then it is easy to see that the positive critical point $t_{a,u}$ of $h_{a,u}$ is unique, a nondegenerate global maximum of $h_{a,u}$. Moreover, $tu \in \mathcal{N}_a$ if and only if $t = t_{a,u}$. Directly calculating $t_{a,u}$ in terms of a and u yields that the mapping $\mathcal{U}_a^+ \to (0,\infty)$, $u \mapsto t_{a,u}$ is continuous. Hence the map $\mathcal{S}_a^+ \to \mathcal{N}_a$, $u \mapsto t_{a,u}u$ is a homeomorphism, with inverse $u \mapsto u/\|u\|$.

If $u \in \mathcal{N}_a$ then

$$\|u\|^2 = \int_{\mathbb{R}^N} a|u|^p \le \int_{\mathbb{R}^N} a^+|u|^p \le \mathcal{C}_p |a^+|_\infty \|u\|^p.$$

Hence there is $C > 0$, independent of a, such that

$$\inf_{u \in \mathcal{N}_a} \|u\| \ge C|a^+|_\infty^{-1/(p-2)} \tag{2.3}$$

and

$$\inf_{u \in \mathcal{N}_a} J_a(u) \ge \left(\frac{1}{2} - \frac{1}{p}\right) C^2 |a^+|_\infty^{-2/(p-2)}. \tag{2.4}$$

Consequently, $\mathcal{N}_a$ is a closed subset of H^1. It is standard to show that $\mathcal{N}_a$ is a submanifold of H^1 of class C^2 and that, denoting by $\overline{J}_a$ the restriction of J_a to $\mathcal{N}_a$, the critical points of $\overline{J}_a$ coincide with the nontrivial (i.e. nonzero) critical points of J_a. It follows as in the definite superlinear case that a ground state cannot change sign and may be taken to be a positive function.

Recall that a sequence $(u_n) \subseteq H^1$ is called a Palais-Smale sequence for J_a (a (PS)-sequence in short) if $(J_a(u_n))_n$ is bounded and $\mathrm{D}J_a(u_n) \to 0$. Moreover, if (u_n) is a (PS)-sequence for J_a and $J_a(u_n) \to c$ then (u_n) is called a $(\mathrm{PS})_c$-sequence for J_a. Since the nonlinearity in (1.1) is homogeneous in u it is standard to show that (PS)-sequences for J_a are bounded in H^1.

Lemma 2.2. *Any* $(\mathrm{PS})_c$*-sequence for* $\overline{J}_a$ *is a* $(\mathrm{PS})_c$*-sequence for* J_a*.*

Proof. Suppose that $(u_n) \subseteq \mathcal{N}_a$ is a $(\mathrm{PS})_c$-sequence for $\overline{J}_a$, for some $c \in \mathbb{R}$. Define the functional $K_a\colon H^1 \to \mathbb{R}$ by $K_a(u) := \mathrm{D}J_a(u)u$. Then $\mathcal{N}_a = K_a^{-1}(0)\backslash\{0\}$. Since $J_a(u_n) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2$, $\|u_n\|$ is bounded and hence there is $C > 0$ such that

$$\frac{\|\nabla K_a(u_n)\|}{\|u_n\|} \le C \qquad \text{for all } n. \tag{2.5}$$

Moreover, for all n

$$\langle \nabla K_a(u_n), u_n \rangle = (2-p)\|u_n\|^2 < 0 \tag{2.6}$$

and

$$\nabla \overline{J}_a(u_n) = \nabla J_a(u_n) - \frac{\langle \nabla J_a(u_n), \nabla K_a(u_n)\rangle \nabla K_a(u_n)}{\|\nabla K_a(u_n)\|^2}. \tag{2.7}$$

Since $u_n \in \mathcal{N}_a$, $\langle \nabla J_a(u_n), u_n \rangle = 0$, i.e. $\nabla J_a(u_n) \perp u_n$. This implies

$$\|\nabla K_a(u_n)\|^2 \geq \left(\frac{\langle \nabla K_a(u_n), u_n\rangle}{\|u_n\|}\right)^2 + \left(\frac{\langle \nabla K_a(u_n), \nabla J_a(u_n)\rangle}{\|\nabla J_a(u_n)\|}\right)^2 \tag{2.8}$$

for all n. It follows that

$$\begin{aligned}
&\|\nabla \overline{J}_a(u_n)\| \cdot \|\nabla J_a(u_n)\| \\
&\qquad \geq \langle \nabla \overline{J}_a(u_n), \nabla J_a(u_n)\rangle \\
&\qquad = \frac{\|\nabla J_a(u_n)\|^2}{\|\nabla K_a(u_n)\|^2}\left(\|\nabla K_a(u_n)\|^2 - \left(\frac{\langle \nabla J_a(u_n), \nabla K_a(u_n)\rangle}{\|\nabla J_a(u_n)\|}\right)^2\right) && \text{by (2.7)} \\
&\qquad \geq \frac{\|\nabla J_a(u_n)\|^2}{\|\nabla K_a(u_n)\|^2}\left(\frac{\langle \nabla K_a(u_n), u_n\rangle}{\|u_n\|}\right)^2 && \text{by (2.8)} \\
&\qquad = \frac{\|\nabla J_a(u_n)\|^2}{\|\nabla K_a(u_n)\|^2}(2-p)^2\|u_n\|^2 && \text{by (2.6)} \\
&\qquad \geq C\|\nabla J_a(u_n)\|^2 && \text{by (2.5).}
\end{aligned}$$

Hence $\|\nabla J_a(u_n)\| \to 0$ and (u_n) is a $(\mathrm{PS})_c$-sequence for J_a. □

3 The Splitting Lemma

In this section we just assume that $a \in L^\infty$. Since in general J_a does not satisfy the (PS)-condition at all levels due to the unboundedness of the domain, the main tool to obtain information about minimizing sequences for $\overline{J}_a$ if $a^+ \neq 0$ is a so-called *Splitting Lemma*. It serves to analyze the possible loss of compactness at infinity and allows, in combination with energy estimates, to regain compactness for these sequences.

A map $\mathcal{F} \colon X \to Y$ between Banach spaces X, Y is said to *BL-split along weakly convergent sequences* (or simply to *BL-split*) if for any sequence $(u_n) \subseteq X$ such that $u_n \rightharpoonup u$ in X it holds true that $\mathcal{F}(u_n) - \mathcal{F}(u_n - u) \to \mathcal{F}(u)$ in Y. A similar notion was introduced (without giving it a name) by Brézis and Lieb [8]. Writing $f(t) := |t|^{p-2}t$ and $F(t) := \frac{1}{p}|t|^p$, it is well known that the maps $u \mapsto \|u\|^2$, $u \mapsto F(u)$ and $u \mapsto f(u)$ BL-split as maps from H^1 into $\mathbb{R}$, L^1 and $L^{p/(p-1)}$, respectively (cf. [1, Lemma 6.3]). It follows that if $(u_n) \subseteq H^1$

and $(y_n) \subseteq \mathbb{R}^N$ are such that $\tau_{-y_n} u_n \rightharpoonup u_0$ in H^1 then

$$\int_{\mathbb{R}^N} |(\tau_{-y_n} a)(F(\tau_{-y_n} u_n) - F(\tau_{-y_n} u_n - u_0) - F(u_0))| \\ \leq |a|_\infty \int_{\mathbb{R}^N} |F(\tau_{-y_n} u_n) - F(\tau_{-y_n} u_n - u_0) - F(u_0)| = o(1)$$

and

$$\int_{\mathbb{R}^N} |(\tau_{-y_n} a)(f(\tau_{-y_n} u_n) - f(\tau_{-y_n} u_n - u_0) - f(u_0))|^{p/(p-1)} \\ \leq |a|_\infty^{p/(p-1)} \int_{\mathbb{R}^N} |f(\tau_{-y_n} u_n) - f(\tau_{-y_n} u_n - u_0) - f(u_0)|^{p/(p-1)} = o(1).$$

Since $\|\cdot\|^2$ BL-splits on H^1 it follows that

$$J_{\tau_{-y_n} a}(\tau_{-y_n} u_n) - J_{\tau_{-y_n} a}(\tau_{-y_n} u_n - u_0) - J_{\tau_{-y_n} a}(u_0) = o(1) \tag{3.1}$$

and

$$\mathrm{D}J_{\tau_{-y_n} a}(\tau_{-y_n} u_n) - \mathrm{D}J_{\tau_{-y_n} a}(\tau_{-y_n} u_n - u_0) - \mathrm{D}J_{\tau_{-y_n} a}(u_0) = o(1). \tag{3.2}$$

Definition 3.1. A sequence $(u_n) \subseteq H^1$ is said to *vanish* if $\tau_{x_n} u_n \rightharpoonup 0$ in H^1 for every sequence $(x_n) \subseteq \mathbb{R}^N$.

If (u_n) vanishes in the above sense then (u_n) is bounded in H^1 and $\tau_{x_n} u_n \to 0$ in L^2_{loc} for every sequence $(x_n) \subseteq \mathbb{R}^N$. Hence

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^N} \int_{B_R(x)} |u_n|^2 = 0$$

is satisfied for every $R > 0$, and Lions' Vanishing Lemma [38, Lemma I.1.] implies that $u_n \to 0$ in L^q for every $q \in (2, 2^*)$.

Definition 3.2. If $(u_n) \subseteq H^1$ then we say that $(\mathrm{D}J_a(u_n))$ vanishes in H^{-1} if

$$\mathrm{D}J_{\tau_{x_n} a}(\tau_{x_n} u_n) \overset{\mathrm{w}^*}{\to} 0$$

for every sequence $(x_n) \subseteq \mathbb{R}^N$.

In other words, $(\mathrm{D}J_a(u_n))$ vanishes in H^{-1} if and only if $(\nabla J_a(u_n))$ vanishes in H^1.

Lemma 3.3. *Suppose that $(u_n) \subseteq H^1$, $(y_n) \subseteq \mathbb{R}^N$, and $a^* \in L^\infty$ are such that $(\mathrm{D}J_a(u_n))$ vanishes in H^{-1}, $\tau_{-y_n} u_n \rightharpoonup u_0$ in H^1, and $\tau_{-y_n} a \overset{\mathrm{w}^*}{\to} a^*$. Define $v_n := u_n - \tau_{y_n} u_0$. Then*

$$J_a(u_n) - J_a(v_n) \to J_{a^*}(u_0), \tag{3.3}$$

$$\|u_n\|^2 - \|v_n\|^2 \to \|u_0\|^2, \tag{3.4}$$

$$\mathrm{D}J_{a^*}(u_0) = 0, \tag{3.5}$$

and $(\mathrm{D}J_a(v_n))$ vanishes in H^{-1}.

Proof. Since $\tau_{-y_n} a \overset{\mathrm{w}^*}{\to} a^*$, it holds true that

$$J_{a^*}(u_0) = J_{\tau_{-y_n} a}(u_0) + o(1).$$

Combining this with (3.1) we obtain (3.3). Since $\|\cdot\|^2$ BL-splits, (3.4) follows.

To show (3.5) let $v \in H^1$. Since $\tau_{-y_n} u_n \to u_0$ in L^p_{loc}, $f(\tau_{-y_n} u_n)v \to f(u_0)v$ in L^1. Hence

$$\begin{aligned}
\mathrm{D}J_{a^*}(u_0)v &= \langle u_0, v\rangle - \int_{\mathbb{R}^N} \tau_{-y_n} a f(u_0)v + o(1) \\
&= \langle \tau_{-y_n} u_n, v\rangle - \int_{\mathbb{R}^N} \tau_{-y_n} a f(\tau_{-y_n} u_n)v + o(1) \\
&= \mathrm{D}J_{\tau_{-y_n} a}(\tau_{-y_n} u_n)v + o(1) \\
&= o(1),
\end{aligned}$$

where we have used that $(\mathrm{D}J_a(u_n))$ vanishes in H^{-1}. Since v was arbitrarily chosen in H^1, (3.5) holds true.

To prove that $(\mathrm{D}J_a(v_n))$ vanishes in H^{-1}, suppose that $(x_n) \subseteq \mathbb{R}^N$ and $v \in H^1$. If $\mathrm{D}J_{\tau_{x_n} a}(\tau_{x_n} v_n)v \to 0$ were not true we could pass to a subsequence such that

$$\liminf_{n\to\infty} |\mathrm{D}J_{\tau_{x_n} a}(\tau_{x_n} v_n)v| > 0 \tag{3.6}$$

and such that either $|x_n + y_n| \to \infty$ or $x_n + y_n \to -\xi$ for some $\xi \in \mathbb{R}^N$. In the first case it would follow that $\tau_{-x_n-y_n} v \rightharpoonup 0$ in H^1 and hence $f(u_0)\tau_{-x_n-y_n} v \to 0$ in L^1. Therefore,

$$\mathrm{D}J_{\tau_{-y_n} a}(u_0)\tau_{-x_n-y_n} v = o(1). \tag{3.7}$$

Using (3.2), (3.7), and the fact that $(\mathrm{D}J_a(u_n))$ vanishes in H^{-1} we would obtain

$$\begin{aligned}
\mathrm{D}J_{\tau_{x_n} a}(\tau_{x_n} v_n)v &= \mathrm{D}J_{\tau_{-y_n} a}(\tau_{-y_n} v_n)\tau_{-x_n-y_n} v \\
&= \mathrm{D}J_{\tau_{-y_n} a}(\tau_{-y_n} u_n)\tau_{-x_n-y_n} v - \mathrm{D}J_{\tau_{-y_n} a}(u_0)\tau_{-x_n-y_n} v + o(1) \\
&= \mathrm{D}J_{\tau_{-y_n} a}(\tau_{-y_n} u_n)\tau_{-x_n-y_n} v + o(1) \\
&= \mathrm{D}J_{\tau_{x_n} a}(\tau_{x_n} u_n)v + o(1) \\
&= o(1),
\end{aligned}$$

in contradiction with (3.6). In the second case we would obtain, using that translation is continuous in H^1 and in L^q for $q \subseteq [1, \infty)$:

$$\begin{aligned}
\mathrm{D}J_{\tau_{x_n} a}(\tau_{x_n} v_n)v &= \mathrm{D}J_{\tau_{-y_n} a}(\tau_{-y_n} v_n)\tau_\xi v + o(1) \\
&= \mathrm{D}J_{\tau_{-y_n} a}(\tau_{-y_n} u_n)\tau_\xi v - \mathrm{D}J_{\tau_{-y_n} a}(u_0)\tau_\xi v + o(1) && \text{by (3.2)} \\
&= -\mathrm{D}J_{\tau_{-y_n} a}(u_0)\tau_\xi v + o(1) && \text{since } (\mathrm{D}J_a(u_n)) \text{ vanishes} \\
&= -\mathrm{D}J_{a^*}(u_0)\tau_\xi v + o(1) \\
&= o(1) && \text{by (3.5)},
\end{aligned}$$

contradicting (3.6). We have therefore proved that $(\mathrm{D}J_a(v_n))$ vanishes in H^{-1}. □

Lemma 3.4 (Splitting Lemma). *Let (u_n) be a Palais-Smale sequence for J_a at the level $c \in \mathbb{R}$. Then either $u_n \to 0$ in H^1 or, after passing to a subsequence, there are $k \in \mathbb{N}$, sequences $(y_n^i)_n \subseteq \mathbb{R}^N$, functions $a^i \in L^\infty$, and functions $u^i \in H^1\backslash\{0\}$ $(i = 1, \dots, k)$ such that each u^i is a critical point of J_{a^i}, and such that the following hold true:*

$$\left| u_n - \sum_{i=1}^{k} \tau_{y_n^i} u^i \right|_p \to 0, \tag{3.8}$$

$$c \geq \sum_{i=1}^{k} J_{a^i}(u^i), \tag{3.9}$$

$$\tau_{-y_n^i} a \overset{\mathrm{w}^*}{\to} a^i, \tag{3.10}$$

and

$$|y_n^i - y_n^j| \to \infty \qquad (i \neq j,\ n \to \infty). \tag{3.11}$$

Proof. Since (u_n) is a (PS)-sequence for J_a, it is bounded in H^1 and $(\mathrm{D}J_a(u_n))$ vanishes in H^{-1}.

If (u_n) vanishes then $|u_n|_p \to 0$ (see above). Since $\mathrm{D}J_a(u_n)u_n = o(1)$, also $\|u_n\| \to 0$.

If (u_n) does not vanish then there exist $u^1 \in H^1\backslash\{0\}$ and a sequence $(y_n^1) \subset \mathbb{R}^N$ such that, after passing to a subsequence and writing $u_n^1 := u_n$, $\tau_{-y_n^1} u_n^1 \rightharpoonup u^1$. By the compactness of $\mathcal{P}$ we may also assume that $\tau_{-y_n^1} a \overset{\mathrm{w}^*}{\to} a^1 \in L^\infty$. Now we define $u_n^2 := u_n^1 - \tau_{y_n^1} u^1$, so $\tau_{-y_n^1} u_n^2 \rightharpoonup 0$. Lemma 3.3 assures that

$$J_a(u_n^1) - J_a(u_n^2) \to J_{a^1}(u^1),$$
$$\|u_n^1\|^2 - \|u_n^2\|^2 \to \|u^1\|^2,$$
$$\mathrm{D}J_{a^1}(u^1) = 0$$

and $(\mathrm{D}J_a(u_n^2))$ vanishes in H^{-1}. If (u_n^2) vanishes, then $|u_n^2|_p \to 0$ and hence

$$|u_n^1 - \tau_{y_n^1} u^1|_p \to 0.$$

Otherwise, there exist $a^2 \in L^\infty$, $u^2 \in H^1\backslash\{0\}$, and a sequence $(y_n^2) \subset \mathbb{R}^N$ such that, after passing to a subsequence, $\tau_{-y_n^2} a \overset{\mathrm{w}^*}{\to} a^2$ and $\tau_{-y_n^2} u_n^2 \rightharpoonup u^2$. Since $\tau_{-y_n^1} u_n^2 \rightharpoonup 0$, $|y_n^1 - y_n^2| \to \infty$.

Proceeding in this way, inductively we obtain sequences (y_n^i), functions $a^i \in L^\infty$, and functions $u^i \in H^1\backslash\{0\}$, for $i = 1, 2, 3, \dots$. Note that since u^i is a nontrivial critical point of J_{a^i} for $i \geq 1$, necessarily $(a^i)^+ \neq 0$. On the other hand, $|(a^i)^+|_\infty \leq |a|_\infty$. Hence $u^i \in \mathcal{N}_{a^i}$ for every i and by (2.3) there is $C > 0$, independent of i, such that $\|u^i\| \geq C$. For every j we have

$$0 \leq \|u_n^{j+1}\|^2 = \|u_n\|^2 - \sum_{i=1}^{j} \|u^i\|^2 + o(1),$$

so by the lower positive bound for $\|u^i\|$ and since (u_n) is bounded in H^1 the process must stop after a finite number of iterations. Therefore, there is $k \in \mathbb{N}$ such that (u_n^{k+1}) vanishes,

$$|u_n^{k+1}|_p \to 0 \tag{3.12}$$

and (3.8) holds true.

Similarly, we have

$$-\int_{\mathbb{R}^N} a|u_n^{k+1}|^p \le J_a(u_n^{k+1}) = J_a(u_n) - \sum_{i=1}^{k} J_{a^i}(u^i) + o(1),$$

so (3.9) is a consequence of (3.12) and of $c = \lim_{n\to\infty} J_a(u_n)$.

The remaining properties (3.10) and (3.11) were already proved along the way. □

4 Energy Estimates and the Existence of a Ground State

In this section we assume $a \in L^\infty$, define $\bar{a}$ as in (1.5), and assume (A).

The technique used in the following lemma is well known, see, e.g., [25]:

Lemma 4.1. *Let $a_1, a_2 \in L^\infty$ be such that $a_1 \ge a_2$. Then $c_{a_1} \le c_{a_2}$. If in addition $a_1 \ne a_2$ and J_{a_2} has a ground state then $c_{a_1} < c_{a_2}$.*

Proof. If $a_2^+ = 0$ then there is nothing to prove. Suppose therefore that $a_2^+ \ne 0$ and that $u \in \mathcal{N}_{a_2}$. It follows that $\int_{\mathbb{R}^N} a_1(x)|u|^p \ge \int_{\mathbb{R}^N} a_2(x)|u|^p > 0$, and we define

$$t := \left(\frac{\int_{\mathbb{R}^N} a_2(x)|u|^p}{\int_{\mathbb{R}^N} a_1(x)|u|^p} \right)^{\frac{1}{p-2}} \le 1.$$

Then

$$\mathrm{D}J_{a_1}(tu)[tu] = t^2 \left(\|u\|^2 - t^{p-2} \int_{\mathbb{R}^N} a_1(x)|u|^p \right) = t^2 \mathrm{D}J_{a_2}(u)u = 0,$$

and $tu \in \mathcal{N}_{a_1}$. On the other hand,

$$J_{a_2}(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p} \int_{\mathbb{R}^N} a_2(x)|u|^p = \left(\frac{1}{2} - \frac{1}{p} \right) \|u\|^2 \ge \left(\frac{1}{2} - \frac{1}{p} \right) \|tu\|^2 = J_{a_1}(tu) \ge c_{a_1}.$$

It follows that $c_{a_2} = \inf_{u \in \mathcal{N}_{a_2}} J_{a_2}(u) \ge c_{a_1}$.

If $a_1 \ne a_2$ and u is a ground state of J_{a_2} then $|u| > 0$. Using the same definition for t as above we obtain $t < 1$. It follows that $c_{a_2} = J_{a_2}(u) > J_{a_1}(tu) \ge c_{a_1}$. □

We identify any constant $b \in \mathbb{R}$ with the constant function $b \in L^\infty$. The next lemma is easily verified:

Lemma 4.2. *Let $b_1, b_2 > 0$. Then u is a ground state of J_{b_1} if and only if $(b_1/b_2)^{1/(p-2)}u$ is a ground state for J_{b_2}, and*

$$J_{b_2}((b_1/b_2)^{1/(p-2)}u) = (b_1/b_2)^{2/(p-2)} J_{b_1}(u).$$

It is well known that there exists a positive and radially symmetric ground state U_1 for J_1 [37]. This could also be proved applying the Splitting Lemma 3.4 to a minimizing sequence for $\overline{J}_1$, see the proof of Theorem 1.1. Define, for every $b > 0$, the function $U_b := (1/b)^{1/(p-2)} U_1$. By Lemma 4.2, U_b is a ground state of J_b and it follows that

$$c_b = J_b(U_b) = (1/b)^{2/(p-2)} J_1(U_1) = (1/b)^{2/(p-2)} c_1.$$

Fix $\varepsilon > 0$ such that

$$0 < \gamma < 2(1-\varepsilon), \tag{4.1}$$

where $\gamma \in (0,2)$ is given by (A), and pick a cut-off function $\chi \in C^\infty(\mathbb{R}^N)$ with $0 \le \chi \le 1$, $\chi(x) = 1$ if $|x| \le 1-\varepsilon$, and $\chi(x) = 0$ if $|x| \ge 1$. For $u \in H^1$ and $R > 0$ define $\Lambda_R u \in H^1$ by

$$(\Lambda_R u)(x) := \chi\left(\frac{x}{R}\right) u(x),$$

for all $x \in \mathbb{R}^N$. Observe that

$$\Lambda_R u \to u \in H^1 \qquad \text{as } R \to \infty. \tag{4.2}$$

Then [19, Lemma 2] states that for all $b, s > 0$

$$\int_{\mathbb{R}^N} \left| |\nabla U_b|^2 - |\nabla \Lambda_R U_b|^2 \right| = O(e^{-2(1-\varepsilon)R}), \tag{4.3}$$

and

$$\int_{|x| \ge R} |U_b|^s dx = O(e^{-sR}) \tag{4.4}$$

as $R \to \infty$.

Proposition 4.3. *It holds true that $c_a < c_{\bar{a}}$.*

Proof. We consider the three cases from condition (A) separately. If (i) $\bar{a} \le 0$ then $c_{\bar{a}} = \infty$. Since $a^+ \ne 0$, $c_a < \infty$ and there is nothing to prove. Therefore we may assume for the remaining cases that $\bar{a} > 0$. If (ii) $a \ge \bar{a}$ then $a \ne \bar{a}$, since $\bar{a} > -\infty$ implies that $\mathcal{B} \ne \varnothing$. Moreover, Lemma 4.1 implies that $c_a < c_{\bar{a}}$ because $J_{\bar{a}}$ has a ground state (see Lemma 4.1). In the remaining case (iii) of (A), inspired by [19], we claim first that there is $D > 0$ such that

$$J_a(t\tau_{z_n}\Lambda_{R_n}U_{\bar{a}}) \le c_{\bar{a}} - De^{-\gamma R_n} \qquad \text{for large } n \text{ and all } t \ge 0. \tag{4.5}$$

To see this, note that $\tau_{-z_n} a \overset{w^*}{\to} \bar{a}$ and (4.2) imply that

$$\int_{\mathbb{R}^N} a\tau_{z_n} |\Lambda_{R_n} U_{\bar{a}}|^p = \int_{\mathbb{R}^N} (\tau_{-z_n} a) U_{\bar{a}}^p + o(1) = \int_{\mathbb{R}^N} \bar{a} U_{\bar{a}}^p + o(1) \tag{4.6}$$

and hence

$$\lim_{n\to\infty} J_a(t\tau_{z_n}\Lambda_{R_n}U_{\bar{a}}) = J_{\bar{a}}(tU_{\bar{a}}) \qquad \text{for all } t \ge 0. \tag{4.7}$$

Moreover, by (4.6)

$$\int_{\mathbb{R}^N} a\tau_{z_n}|\Lambda_{R_n}U_{\bar{a}}|^p > 0 \qquad \text{for large } n. \tag{4.8}$$

Therefore, the fibering maps $t \mapsto J_{\bar{a}}(tU_{\bar{a}})$ and $t \mapsto J_a(t\tau_{z_n}\Lambda_{R_n}U_{\bar{a}})$ have, for large n, exactly one critical point in $(0,\infty)$, and it is a global maximum. Moreover, the fibering maps take the value 0 at 0 and tend to $-\infty$ as $t \to \infty$. Recalling that the maximum of $t \mapsto J_{\bar{a}}(tU_{\bar{a}})$ is $c_{\bar{a}}$, these facts imply, together with (4.7), that there are $n_0 \in \mathbb{N}$ and $t_1 > t_0 > 0$ such that

$$\forall n \geq n_0 \; \forall t \in [0,t_0] \cup [t_1,\infty)\colon \; J_a(t\tau_{z_n}\Lambda_{R_n}U_{\bar{a}}) \leq \frac{c_{\bar{a}}}{2}. \tag{4.9}$$

Passing to a subsequence we may assume that (R_n) is an increasing sequence. Using (A) and the positivity of $U_{\bar{a}}$ choose $n_1 \geq n_0$ such that

$$C := \int_{\|x\| \leq (1-\varepsilon)R_{n_1}} \kappa U_{\bar{a}}^p > 0. \tag{4.10}$$

It follows for $n \geq n_1$ that

$$\begin{aligned}
&\int_{\mathbb{R}^N} a|\tau_{z_n}\Lambda_{R_n}U_{\bar{a}}|^p \\
&\qquad \geq \int_{\|x\|\leq R_n} (\tau_{-z_n}a)|\Lambda_{R_n}U_{\bar{a}}|^p && \text{since } a \geq 0 \text{ on } B_{R_n}(z_n) \\
&\qquad \geq \int_{\|x\|\leq R_n} \left(\bar{a} + \kappa e^{-\gamma R_n}\right)|\Lambda_{R_n}U_{\bar{a}}|^p && \text{by (A)} \\
&\qquad \geq \bar{a}\int_{\|x\|\leq(1-\varepsilon)R_n} U_{\bar{a}}^p + e^{-\gamma R_n}\int_{\|x\|\leq(1-\varepsilon)R_{n_1}} \kappa U_{\bar{a}}^p \\
&\qquad \geq \bar{a}\int_{\mathbb{R}^N} U_{\bar{a}}^p + O(e^{-p(1-\varepsilon)R_n}) + e^{-\gamma R_n}\int_{\|x\|\leq(1-\varepsilon)R_{n_1}} \kappa U_{\bar{a}}^p && \text{by (4.4)} \\
&\qquad \geq \bar{a}\int_{\mathbb{R}^N} U_{\bar{a}}^p + O(e^{-p(1-\varepsilon)R_n}) + Ce^{-\gamma R_n} && \text{by (4.10).}
\end{aligned}$$

Together with (4.3) and (4.4) this implies for $n \geq n_1$ and $t \in [t_0, t_1]$

$$\begin{aligned}
J_a(t\tau_{z_n}\Lambda_{R_n}U_{\bar{a}}) &= \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla\tau_{z_n}\Lambda_{R_n}U_{\bar{a}}|^2 + \frac{t^2}{2}\int_{\mathbb{R}^N}\tau_{z_n}(\Lambda_{R_n}U_{\bar{a}})^2 - \frac{t^p}{p}\int_{\mathbb{R}^N} a(x)|\tau_{z_n}\Lambda_{R_n}U_{\bar{a}}|^p \\
&\leq \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla U_{\bar{a}}|^2 + \frac{t^2}{2}\int_{\mathbb{R}^N} U_{\bar{a}}^2 + t^2 O(e^{-2(1-\varepsilon)R_n}) - \frac{t^p}{p}\int_{\mathbb{R}^N} a(x)|\tau_{z_n}\Lambda_{R_n}U_{\bar{a}}|^p \\
&\leq J_{\bar{a}}(tU_{\bar{a}}) + t_1^2 O(e^{-2(1-\varepsilon)R_n}) + t_1^p O(e^{-p(1-\varepsilon)R_n}) - \frac{t_0^p}{p}Ce^{-\gamma R_n} \\
&\leq c_{\bar{a}} + O(e^{-2(1-\varepsilon)R_n}) - \frac{t_0^p}{p}Ce^{-\gamma R_n}.
\end{aligned}$$

By (4.1) and (4.9) the claim (4.5) follows.

Using (4.8) pick $n_0 \in \mathbb{N}$ such that there is $t_0 > 0$ with $t_0\tau_{z_{n_0}}\Lambda_{R_{n_0}}U_{\bar{a}} \in \mathcal{N}_a$ and such that (4.5) holds true for $n = n_0$. It follows that

$$c_a \leq J_a(t_0\tau_{z_{n_0}}\Lambda_{R_{n_0}}U_{\bar{a}}) \leq c_{\bar{a}} - De^{-\gamma R_{n_0}} < c_{\bar{a}}. \qquad \square$$

Proof of Theorem 1.1. Since $a^+ \neq 0$, $\mathcal{N}_a \neq \varnothing$ and $c_a < \infty$. Moreover, $c_a > 0$ by (2.4). Using Ekeland's Lemma we can therefore build a minimizing sequence $(u_n) \subseteq \mathcal{N}_a$ for $\overline{J}_a = J_a|_{\mathcal{N}_a}$ that is also a (PS)-sequence for $\overline{J}_a$ at the level c_a. By Lemma 2.2 (u_n) is a (PS)-sequence for J_a at the level c_a. It cannot happen that $u_n \to 0$ since $J_a(u_n) \to c_a > 0$. Therefore Lemma 3.4 implies the existence of $k \in \mathbb{N}$, functions $a^i \in \overline{\mathcal{A}}$ and nontrivial critical points u^i of J_{a^i} such that

$$c_a \geq \sum_{i=1}^{k} J_{a^i}(u^i).$$

Since the functions u^i are nontrivial critical points of J_{a^i}, $(a^i)^+ \neq 0$ for every i. By (2.4), $J_{a^i}(u^i) > 0$ for every i.

Assume that there is i such that $a^i \in \mathcal{B}$. Since then $a^i \leq \bar{a}$, Lemma 4.1 and Proposition 4.3 imply that $J_{a^i}(u^i) \geq c_{a^i} \geq c_{\bar{a}} > c_a$, a contradiction. Therefore, each a^i belongs to $\mathcal{A}$ and is a translate of a. Hence $J_{a^i}(u^i) \geq c_a$ for every i. This implies that $k = 1$ and that a translate of u^1 is a ground state of J_a. □

5 Tools for the Construction of Examples

Proof of Proposition 1.2. If $\mathcal{B} = \varnothing$ then $\bar{a} = -\infty$ and (A) is satisfied. Assume therefore that $\mathcal{B} \neq \varnothing$. In particular, a is not constant. We claim that

$$\bar{a} \leq \hat{a}. \tag{5.1}$$

To see this, suppose that $b \in \mathcal{B}$. There is $(x_n) \subseteq \mathbb{R}^N$ such that $\tau_{x_n} a \overset{\mathrm{w}^*}{\to} b$. If (x_n) contained a bounded subsequence then, after passing to a subsequence, there would exist $\xi \in \mathbb{R}^N$ such that $x_n \to \xi$ and $\tau_{x_n} a \overset{\mathrm{w}^*}{\to} \tau_\xi a \in \mathcal{A}$. This follows from weak*-continuity of translation in L^∞, which in turn is a consequence of continuity of translation in L^1. On the other hand, since $\mathcal{P}$ is metrizable, $\tau_\xi a = b \notin \mathcal{A}$, a contradiction. Therefore $|x_n| \to \infty$.

Given $\varepsilon > 0$, by Lemma 2.1 there is $\varphi \in S_1 L^1$ such that $\varphi \geq 0$ and

$$\int_{\mathbb{R}^N} b\varphi \geq \operatorname{ess\,sup} b - \frac{\varepsilon}{2}.$$

Take $\widetilde{\psi} \in C_{\mathrm{c}}^\infty$ such that $\widetilde{\psi} \geq 0$ and

$$|\varphi - \widetilde{\psi}|_1 \leq \frac{\varepsilon}{4|b|_\infty}.$$

Set $\psi := \widetilde{\psi}/|\widetilde{\psi}|_1$. Then

$$|\varphi - \psi|_1 \leq \frac{\varepsilon}{2|b|_\infty}$$

and $\psi \in S_1 L^1 \cap C_c^\infty$ satisfies $\psi \geq 0$. We obtain

$$\int_{\mathbb{R}^N} b\psi = \int_{\mathbb{R}^N} b\varphi - \int_{\mathbb{R}^N} b(\varphi - \psi) \geq \int_{\mathbb{R}^N} b\varphi - |b|_\infty |\psi - \varphi|_1 \geq \operatorname{ess\,sup} b - \varepsilon.$$

Suppose that $\operatorname{supp}\psi \subseteq B_R$. Then

$$\begin{aligned}\operatorname{ess\,sup} b - \varepsilon \leq \int_{\mathbb{R}^N} b\psi = \lim_{n\to\infty} \int_{\mathbb{R}^N} (\tau_{x_n} a)\psi \\ \leq \lim_{n\to\infty} \operatorname{ess\,sup} \left(a|_{B_R(-x_n)}\right) \int_{B_R} \psi \leq \lim_{n\to\infty} \operatorname{ess\,sup} \left(a|_{\mathbb{R}^N \setminus B_{|x_n|-R}}\right) = \hat{a}.\end{aligned}$$

Letting $\varepsilon \to 0$ this yields $\operatorname{ess\,sup} b \leq \hat{a}$ and hence (5.1).

We now consider the three subcases of (A), under the assumption $\mathcal{B} \neq \varnothing$. In case (i) we obtain from (5.1) that $\bar{a} \leq \hat{a} \leq 0$. In case (ii) (5.1) implies $\bar{a} \leq \hat{a} \leq a$. And in case (iii) there is a sequence $(z_n) \subseteq \mathbb{R}^N$ such that $\tau_{-z_n} a \overset{w^*}{\to} \hat{a}$, i.e., $\hat{a} \in \overline{\mathcal{A}}$. Since a is not constant, $\hat{a} \in \mathcal{B}$. Therefore $\hat{a} \leq \bar{a}$, which implies together with (5.1) that $\hat{a} = \bar{a}$. In this case, the original condition (A) is satisfied, with $\bar{a}$ instead of $\hat{a}$. □

Proof of Proposition 1.5. Define the linear operator $\Gamma\colon \mathbb{R}^{\mathbb{R}^k} \to \mathbb{R}^{\mathbb{R}^k \times \mathbb{R}^\ell}$ by $(\Gamma u)(x, y) := u(x)$. Suppose that $a \in L^\infty(\mathbb{R}^k)$ and set $a' := \Gamma a$. Moreover, define by $\mathcal{A}'$ and $\mathcal{B}'$ the corresponding sets for a' as in (1.3) and (1.4), and define $\bar{a}'$ as in (1.5).

We claim that

$$\Gamma \text{ restricts to a bijection } \mathcal{B} \to \mathcal{B}'. \tag{5.2}$$

Clearly Γ is injective. We show first that $b' := \Gamma b \in \mathcal{B}'$ if $b \in \mathcal{B}$. There is $(x_n) \subseteq \mathbb{R}^k$ such that $\tau_{x_n} a \overset{w^*}{\to} b$. Since $b \notin \mathcal{A}$, $|x_n| \to \infty$ (see the proof of Proposition 1.2). For any $\varphi \in L^1(\mathbb{R}^{k+\ell})$ the function

$$x \mapsto \int_{\mathbb{R}^\ell} \varphi(x, y)\,\mathrm{d}y$$

is in $L^1(\mathbb{R}^k)$, by Fubini's Theorem. We obtain

$$\begin{aligned}\int_{\mathbb{R}^{k+\ell}} (\tau_{(x_n,0)} a')\varphi = \int_{\mathbb{R}^{k+\ell}} (\tau_{x_n} a)(x)\varphi(x, y)\,\mathrm{d}(x, y) = \int_{\mathbb{R}^k} (\tau_{x_n} a)(x) \int_{\mathbb{R}^\ell} \varphi(x, y)\,\mathrm{d}y\,\mathrm{d}x \\ \to \int_{\mathbb{R}^k} b(x) \int_{\mathbb{R}^\ell} \varphi(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_{\mathbb{R}^{k+\ell}} b'\varphi. \quad (5.3)\end{aligned}$$

Hence $\tau_{(x_n,0)} a' \overset{w^*}{\to} b'$ and $b' \in \overline{\mathcal{A}'}$. If $b' \in \mathcal{A}'$ were true, there would exist $(x_0, y_0) \in \mathbb{R}^{k+\ell}$ such that $\tau_{(x_0,y_0)} b' = a'$. For all $(x, y) \in \mathbb{R}^{k+\ell}$ this would imply

$$b(x - x_0) = b'(x - x_0, y - y_0) = a'(x, y) = a(x)$$

and therefore $\tau_{x_0} b = a$. But this would contradict $b \notin \mathcal{A}$. Therefore $b' \in \mathcal{B}'$. We have shown that $\Gamma(\mathcal{B}) \subseteq \mathcal{B}'$.

To show surjectivity, suppose that $b' \in \mathcal{B}'$. There is a sequence $((x_n, y_n)) \subseteq \mathbb{R}^{k+\ell}$ such that $\tau_{(x_n,y_n)}a' \xrightarrow{\text{w}^*} b'$. As before, $|(x_n, y_n)| \to \infty$ since $b' \notin \mathcal{A}'$. Moreover, since $a'(x, y) = a(x)$ for all (x, y), we can assume that $y_n = 0$ for all n, so $\tau_{(x_n,0)}a' \xrightarrow{\text{w}^*} b'$ and $|x_n| \to \infty$.

Suppose that $(x^*, y), (x^*, y') \in \mathbb{R}^{k+\ell}$ are Lebesgue points of b'. We have

$$\tau_{(0,y-y')}b' = \operatorname*{w^*\text{-}lim}_{n\to\infty} \tau_{(x_n,y-y')}a' = \operatorname*{w^*\text{-}lim}_{n\to\infty} \tau_{(x_n,0)}a' = b'.$$

Therefore

$$\begin{aligned} b'(x^*, y) = \lim_{r\to 0} \frac{1}{|B_r|} \int_{B_r((x^*,y))} b' = \lim_{r\to 0} \frac{1}{|B_r|} \int_{B_r((x^*,y))} \tau_{(0,y-y')}b' \\ = \lim_{r\to 0} \frac{1}{|B_r|} \int_{B_r((x^*,y'))} b' = b'(x^*, y'). \end{aligned}$$

Since the complement of the set of Lebesgue points of b' has zero measure, this shows that b' is independent of the second argument and that there is $b \in L^\infty$ such that $b' = \Gamma b$. We need to show that $b \in \mathcal{B}$. Suppose that $\varphi \in L^1(\mathbb{R}^k)$. Pick any $\psi \in L^1(\mathbb{R}^\ell)$ such that $\int \psi = 1$. Then the map ϑ given by $\vartheta(x, y) := \varphi(x)\psi(y)$ is in $L^1(\mathbb{R}^{k+\ell})$, by Tonelli's Theorem. Moreover

$$\int_{\mathbb{R}^k} (\tau_{x_n}a)\varphi = \int_{\mathbb{R}^k} (\tau_{x_n}a)(x) \int_{\mathbb{R}^\ell} \varphi(x)\psi(y)\,\mathrm{d}y\,\mathrm{d}x = \int_{\mathbb{R}^{k+\ell}} (\tau_{(x_n,0)}a')\vartheta \to \int_{\mathbb{R}^{k+\ell}} b'\vartheta = \int_{\mathbb{R}^k} b\varphi.$$

Hence $b \in \overline{\mathcal{A}}$. If b belonged to $\mathcal{A}$ then clearly b' would belong to $\mathcal{A}'$, a contradiction. Therefore $b \in \mathcal{B}$ and we have proved (5.2).

We now consider a number of cases. If $\mathcal{B} = \varnothing$ then $\mathcal{B}' = \Gamma(\mathcal{B}) = \varnothing$ and $\bar{a}' = -\infty \le 0$, that is, a' satisfies (A). We therefore assume now that $\mathcal{B} \neq \varnothing$. It follows that

$$\bar{a}' = \sup_{b'\in\mathcal{B}'} \operatorname{ess\,sup} b' = \sup_{b\in\mathcal{B}} \operatorname{ess\,sup} \Gamma b = \sup_{b\in\mathcal{B}} \operatorname{ess\,sup} b = \bar{a}.$$

In case (A)(i) we obtain $\bar{a}' = \bar{a} \le 0$. In case (A)(ii) we have $\bar{a} \le a$. This implies $\bar{a}' = \bar{a} \le a'$. And in case (A)(iii) there are sequences $(z_n) \subseteq \mathbb{R}^k$ and $R_n \to \infty$ such that $\tau_{-z_n}a \xrightarrow{\text{w}^*} \bar{a}$ and

$$\forall x \in B_{R_n}(z_n)\colon a(x) \ge \bar{a} + \kappa(x - z_n)\mathrm{e}^{-\gamma R_n}.$$

Put $z_n' := (z_n, 0)$ and define $\kappa' := \Gamma\kappa$. As in (5.3) $\tau_{-z_n}a' \xrightarrow{\text{w}^*} \bar{a} = \bar{a}'$. If $(x, y) \in B^{k+\ell}_{R_n}(z_n')$ then $x \in B^k_{R_n}(z_n)$ and hence

$$a'(x, y) = a(x) \ge \bar{a} + \kappa(x - z_n)\mathrm{e}^{-\gamma R_n} = \bar{a}' + \kappa'((x, y) - z_n')\mathrm{e}^{-\gamma R_n}.$$

In all cases a' satisfies (A). □

References

[1] N. Ackermann, *A nonlinear superposition principle and multibump solutions of periodic Schrödinger equations*, J. Funct. Anal. **234** (2006), no. 2, 277–320. MR MR2216902

[2] A. Ambrosetti, V. Felli, and A. Malchiodi, *Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity*, J. Eur. Math. Soc. (JEMS) **7** (2005), no. 1, 117–144. MR MR2120993 (2006f:35049)

[3] A. Bahri and Y.Y. Li, *On a min-max procedure for the existence of a positive solution for certain scalar field equations in* $\mathbf{R}^N$, Rev. Mat. Iberoamericana **6** (1990), no. 1-2, 1–15. MR 1086148 (92b:35054)

[4] A. Bahri and P.L. Lions, *On the existence of a positive solution of semilinear elliptic equations in unbounded domains*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), no. 3, 365–413. MR MR1450954 (98k:35047)

[5] T. Bartsch and M. Willem, *Infinitely many radial solutions of a semilinear elliptic problem on* $\mathbf{R}^N$, Arch. Rational Mech. Anal. **124** (1993), no. 3, 261–276. MR 1237913

[6] V. Benci and G. Cerami, *Positive solutions of some nonlinear elliptic problems in exterior domains*, Arch. Rational Mech. Anal. **99** (1987), no. 4, 283–300. MR 88f:35014

[7] H. Berestycki and P.L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. **82** (1983), no. 4, 313–345. MR 84h:35054a

[8] H. Brézis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), no. 3, 486–490. MR 699419 (84e:28003)

[9] K.J. Brown and Y. Zhang, *The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function*, J. Differential Equations **193** (2003), no. 2, 481–499. MR 1998965 (2004i:35069)

[10] D.M. Cao, *Multiple solutions of a semilinear elliptic equation in* $\mathbf{R}^N$, Ann. Inst. H. Poincaré Anal. Non Linéaire **10** (1993), no. 6, 593–604. MR 1253603 (94m:35100)

[11] G. Cerami, *Some nonlinear elliptic problems in unbounded domains*, Milan J. Math. **74** (2006), 47–77. MR 2278729 (2007k:35128)

[12] ———, *Existence and multiplicity results for some scalar fields equations*, Analysis and topology in nonlinear differential equations, Progr. Nonlinear Differential Equations Appl., vol. 85, Birkhäuser/Springer, Cham, 2014, pp. 207–230. MR 3330731

[13] G. Cerami and R. Molle, *On some Schrödinger equations with non regular potential at infinity*, Discrete Contin. Dyn. Syst. **28** (2010), no. 2, 827–844. MR 2644767 (2011i:35090)

[14] G. Cerami, R. Molle, and D. Passaseo, *Multiplicity of positive and nodal solutions for scalar field equations*, J. Differential Equations **257** (2014), no. 10, 3554–3606. MR 3260234

[15] G. Cerami and A. Pomponio, *On some scalar field equations with competing coefficients*, Preprint, August 2015.

[16] J. Chabrowski, *Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents*, Calc. Var. Partial Differential Equations **3** (1995), no. 4, 493–512. MR 97j:35029

[17] J. Chen and S. Li, *Existence and multiplicity of nontrivial solutions for an elliptic equation on $\mathbb{R}^N$ with indefinite linear part*, Manuscripta Math. **111** (2003), no. 2, 221–239. MR 1989424 (2004g:35081)

[18] Y.H. Cheng and T.F. Wu, *Existence and multiplicity of positive solutions for indefinite semilinear elliptic problems in $\mathbb{R}^n$*, Electron. J. Differential Equations (2014), No. 102, 27. MR 3194008

[19] M. Clapp and T. Weth, *Multiple solutions of nonlinear scalar field equations*, Comm. Partial Differential Equations **29** (2004), no. 9-10, 1533–1554. MR MR2103845 (2005h:35087)

[20] D.G. Costa, Y. Guo, and M. Ramos, *Existence and multiplicity results for nonlinear elliptic problems in $\mathbb{R}^N$ with an indefinite functional*, Electron. J. Differential Equations (2002), No. 25, 15. MR 1884994 (2002m:35047)

[21] D.G. Costa and H. Tehrani, *Existence of positive solutions for a class of indefinite elliptic problems in $\mathbb{R}^N$*, Calc. Var. Partial Differential Equations **13** (2001), no. 2, 159–189. MR 2002g:35054

[22] D.G. Costa and H. Tehrani, *Existence and multiplicity results for a class of Schrödinger equations with indefinite nonlinearities*, Adv. Differential Equations **8** (2003), no. 11, 1319–1340. MR 2016649 (2004k:35112)

[23] D.G. Costa, H. Tehrani, and M. Ramos, *Non-zero solutions for a Schrödinger equation with indefinite linear and nonlinear terms*, Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), no. 2, 249–258. MR MR2056283 (2005c:35088)

[24] V. Coti Zelati and P.H. Rabinowitz, *Homoclinic type solutions for a semilinear elliptic PDE on $\mathbf{R}^n$*, Comm. Pure Appl. Math. **45** (1992), no. 10, 1217–1269. MR 93k:35087

[25] W.Y. Ding and W.M. Ni, *On the existence of positive entire solutions of a semilinear elliptic equation*, Arch. Rational Mech. Anal. **91** (1986), no. 4, 283–308. MR 807816 (87b:35056)

[26] T. Dohnal, M. Plum, and W. Reichel, *Surface gap soliton ground states for the nonlinear Schrödinger equation*, Comm. Math. Phys. **308** (2011), no. 2, 511–542. MR 2851151 (2012j:35132)

[27] W. Dong and L. Mei, *Multiple solutions for an indefinite superlinear elliptic problem on* $\mathbb{R}^N$, Nonlinear Anal. **73** (2010), no. 7, 2056–2070. MR 2674184 (2011h:35100)

[28] Y. Du, *Multiplicity of positive solutions for an indefinite superlinear elliptic problem on* $\mathbf{R}^N$, Ann. Inst. H. Poincaré Anal. Non Linéaire **21** (2004), no. 5, 657–672. MR MR2086753 (2005e:35062)

[29] Y. Du and Y. Guo, *Mountain pass solutions and an indefinite superlinear elliptic problem on* $\mathbb{R}^N$, Topol. Methods Nonlinear Anal. **22** (2003), no. 1, 69–92. MR MR2037267 (2005a:35060)

[30] G. Evéquoz and T. Weth, *Entire solutions to nonlinear scalar field equations with indefinite linear part*, Adv. Nonlinear Stud. **12** (2012), no. 2, 281–314. MR 2951719

[31] J. Giacomoni, M. Lucia, and M. Ramaswamy, *Some elliptic semilinear indefinite problems on* $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), no. 2, 333–361. MR 2021681 (2005c:35096)

[32] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. I*, J. Funct. Anal. **74** (1987), no. 1, 160–197. MR 901236 (88g:35169)

[33] Y. Jalilian and A. Szulkin, *Infinitely many solutions for semilinear elliptic problems with sign-changing weight functions*, Appl. Anal. **93** (2014), no. 4, 756–770. MR 3180017

[34] Y. Li, *Remarks on a semilinear elliptic equation on* $\mathbf{R}^n$, J. Differential Equations **74** (1988), no. 1, 34–49. MR 949624 (89h:35112)

[35] Y. Li, Z.Q. Wang, and J. Zeng, *Ground states of nonlinear Schrödinger equations with potentials*, Ann. Inst. H. Poincaré Anal. Non Linéaire **23** (2006), no. 6, 829–837. MR 2271695 (2007g:35057)

[36] X. Lin and X.H. Tang, *Nehari-type ground state positive solutions for superlinear asymptotically periodic Schrödinger equations*, Abstr. Appl. Anal. (2014), Art. ID 607078, 7. MR 3230529

[37] P.L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. I*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no. 2, 109–145. MR 778970 (87e:49035a)

[38] ———, *The concentration-compactness principle in the calculus of variations. The locally compact case. II*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no. 4, 223–283. MR 778974 (87e:49035b)

[39] F. Liu and J. Yang, *Nontrivial solutions of Schrödinger equations with indefinite nonlinearities*, J. Math. Anal. Appl. **334** (2007), no. 1, 627–645. MR 2332581 (2008h:35095)

[40] Z. Liu, J. Su, and T. Weth, *Compactness results for Schrödinger equations with asymptotically linear terms*, J. Differential Equations **231** (2006), no. 2, 501–512. MR 2287894 (2009d:35095)

[41] R. Molle and D. Passaseo, *Multiplicity of solutions of nonlinear scalar field equations*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **26** (2015), no. 1, 75–82. MR 3345323

[42] B. Noris and G. Verzini, *A remark on natural constraints in variational methods and an application to superlinear Schrödinger systems*, J. Differential Equations **254** (2013), no. 3, 1529–1547. MR 2997381

[43] P.H. Rabinowitz, *A note on a semilinear elliptic equation on* $\mathbf{R}^n$, Nonlinear analysis (A. Ambrosetti and A. Marino, eds.), Quaderni, vol. ?, Scuola Norm. Sup., Pisa, 1991, pp. 307–317. MR 94a:35040

[44] ———, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. **43** (1992), no. 2, 270–291. MR MR1162728 (93h:35194)

[45] W. Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987. MR MR924157 (88k:00002)

[46] ———, *Functional analysis*, second ed., International Series in Pure and Applied Mathematics, McGraw-Hill Inc., New York, 1991. MR MR1157815 (92k:46001)

[47] M. Schneider, *Existence and nonexistence of positive solutions of indefinite elliptic problems in* $\mathbb{R}^N$, Adv. Nonlinear Stud. **3** (2003), no. 2, 231–259. MR 1971314 (2004e:35084)

[48] B. Sirakov, *Existence and multiplicity of solutions of semi-linear elliptic equations in* $\mathbf{R}^N$, Calc. Var. Partial Differential Equations **11** (2000), no. 2, 119–142. MR 2002b:35061

[49] C.A. Stuart, *Lectures on the orbital stability of standing waves and application to the nonlinear Schrödinger equation*, Milan J. Math. **76** (2008), 329–399. MR 2465996 (2009k:37165)

[50] A. Szulkin and T. Weth, *Ground state solutions for some indefinite variational problems*, J. Funct. Anal. **257** (2009), no. 12, 3802–3822. MR 2557725 (2010j:35193)

[51] X. Tang, *Non-Nehari manifold method for asymptotically periodic Schrödinger equations*, Sci. China Math. **58** (2015), no. 4, 715–728. MR 3319307

[52] H. Tehrani, *Infinitely many solutions for an indefinite semilinear elliptic problem in* $\mathbf{R}^N$, Adv. Differential Equations **5** (2000), no. 10-12, 1571–1596. MR 1785686 (2001h:35072)

[53] J. Wang, L. Tian, J. Xu, and F. Zhang, *Existence and nonexistence of the ground state solutions for nonlinear Schrödinger equations with nonperiodic nonlinearities*, Math. Nachr. **285** (2012), no. 11-12, 1543–1562. MR 2959974

[54] J. Wei and S. Yan, *Infinitely many positive solutions for the nonlinear Schrödinger equations in* $\mathbb{R}^N$, Calc. Var. Partial Differential Equations **37** (2010), no. 3-4, 423–439. MR 2592980 (2011c:35065)

[55] T.F. Wu, *The existence of multiple positive solutions for a semilinear elliptic equation in* $\mathbb{R}^N$, Nonlinear Anal. **72** (2010), no. 7-8, 3412–3421. MR 2587373 (2011a:35185)

[56] Y. Wu and Y. Huang, *Sign-changing solutions for Schrödinger equations with indefinite supperlinear nonlinearities*, J. Math. Anal. Appl. **401** (2013), no. 2, 850–860. MR 3018032

[57] H. Zhang, J. Xu, and F. Zhang, *Ground state solutions for asymptotically periodic Schrödinger equations with indefinite linear part*, Math. Methods Appl. Sci. **38** (2015), no. 1, 113–122. MR 3291306

[58] F. Zhao, L. Zhao, and Y. Ding, *Existence and multiplicity of solutions for a nonperiodic Schrödinger equation*, Nonlinear Anal. **69** (2008), no. 11, 3671–3678. MR MR2463323

Contact information:

Nils Ackermann: (Corresponding author); Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., 04510 México D.F., Mexico; email: nils@ackermath.info

Julián Chagoya: Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., 04510 México D.F., Mexico