Mapping Degree and Fixed Point Theorems for Nonlinear Operators

Nils Ackermann

SoSe 2019

Contents

1 Introduction

1.1 Images and Fixed Points

A question that often appears is the following: Given a vector space X, a subset $A \subseteq X$, a function $f\colon A \to X$ and a point $y \in X$, is there $x \in A$ such that $f(x) = y$ (a *y-point of f*)? Equivalently, does $y \in \mathcal{R}(f)$ hold true, where $\mathcal{R}(f) := f(A)$ is the *image of f*?

1.1 Examples. (a) Given $g\colon A \to A$ we search for a fixed point of g, that is, $x \in A$ such that $g(x) = x$. Defining $f\colon A \to X$ by $f(x) := g(x) - x$ this problem is equivalent to searching for a zero of f. Note that this equivalence depends on the given structure of the additive group in the vector space X. In general sets we do not have an equivalence of fixed point and zero point problems!

(b) Suppose that $f\colon \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable. We consider an ordinary differential equation with boundary values for functions $u(t)$ that are defined on $[0, 1]$:

$$\begin{cases} \ddot{u}(t) = f(t, u(t), \dot{u}(t)), & t \in (0, 1), \\ u(0) = 0, \\ u(1) = 0. \end{cases} \tag{1.1}$$

The theory of ODEs implies the existence of a continuous function $k(s, t)$ (an *integral kernel*) such that $u(t)$ is a solution of (1.1) if and only if $u \in C^2([0, 1])$ and

$$u(s) = \int_0^1 k(s, t) f(t, u(t), \dot{u}(t)) \,\mathrm{d}t \qquad \text{for all } s \in [0, 1]. \tag{1.2}$$

We set $X := C^2([0, 1])$ and define $F\colon X \to X$ by

$$F(u)(s) := \int_0^1 k(s, t) f(t, u(t), \dot{u}(t)) \,\mathrm{d}t.$$

By (1.2) u is a solution of (1.1) if and only if u is a fixed point of F.

(c) Similarly, we may look for solutions of PDEs: Suppose that $\Omega \subseteq \mathbb{R}^N$ is a bounded domain and consider

$$\begin{aligned} -\Delta u &= f(x, u(x), \nabla u(x)), && x \in \Omega \\ u(x) &= 0, && x \in \partial\Omega. \end{aligned}$$

(d) We may wonder if a problem like those introduced above has more than one fixed point. To address this question we need a more refined theory than that of fixed points.

1.2 The mapping Degree in One Dimension

In one dimension the mapping degree is just another way to express the content of the intermediate value theorem. It serves to illustrate how one comes to define the mapping degree in higher dimensions. Moreover, it will become clear later how the mapping degree is an extension of the intermediate value theorem to higher dimension.

Suppose that $a < b$, $\Omega := (a, b)$ and $f \in C(\overline{\Omega})$ are such that $0 \in \mathbb{R} \backslash f(\partial\Omega)$. We define

$$\deg(f, \Omega, 0) := \begin{cases} 0, & f(a)f(b) > 0, \\ 1, & f(a)f(b) < 0,\ f(a) < 0, \\ -1, & f(a)f(b) < 0,\ f(a) > 0. \end{cases} \tag{1.3}$$

If $y \in \mathbb{R} \backslash f(\partial\Omega)$, then $f(a) - y \neq 0$ and $f(b) - y \neq 0$, that is, $0 \in \mathbb{R} \backslash (f - y)(\partial\Omega)$. We define

$$\deg(f, \Omega, y) := \deg(f - y, \Omega, 0).$$

It is easy to see that in this situation the following hold true:

(i) $\deg(\mathrm{id}, \Omega, y) = 1$ if $y \in \Omega$,

(ii) $\deg(f, \Omega, y) = \deg(f, \Omega_1, y) + \deg(f, \Omega_2, y)$ if $c \in (a, b)$, $\Omega_1 = (a, c)$, $\Omega_2 = (c, b)$ and $y \neq f(c)$.

To address the question of existence of a y-point of f, the following consequence of the intermediate value theorem is fundamental:

(iii) if $\deg(f, \Omega, y) \neq 0$, then f has a y-point in Ω.

Item (ii) gives information about the location of y in Ω: If $f(c) \neq y$, if we know $\deg(f, \Omega, y)$ and $\deg(f, \Omega_1, y)$, and if $\deg(f, \Omega_2, y) = \deg(f, \Omega, y) - \deg(f, \Omega_1, y) \neq 0$, then there is a y-point in Ω_2.

Another property is the *homotopy invariance* of the mapping degree:

(iv) $\deg(h(t, \cdot), \Omega, y(t))$ is independent of t if $h \in C([0, 1] \times \overline{\Omega})$ and $y \in C([0, 1])$ are such that $y(t) \in \mathbb{R} \backslash h(t, \partial\Omega)$ for all $t \in [0, 1]$.

In this situation the function h is called a *homotopy*. One could think of it as a continuous deformation from the function $h(0, \cdot)$ to $h(1, \cdot)$ in $C(\overline{\Omega})$.

The property (iv) is proved as follows: To calculate the degree for every t we set $H(t, x) := h(t, x) - y(t)$ for $(t, x) \in [0, 1] \times \overline{\Omega}$. Then $H(t, x) \neq 0$ for all $(t, x) \in [0, 1] \times \partial\Omega$. Since the function $t \mapsto H(t, a)$ is continuous, it does not change sign, by the intermediate value theorem. The same holds true for the function $t \mapsto H(t, b)$. Therefore, the degree $\deg(H(t, \cdot), \Omega, 0)$ is independent of t because it only depends on the signs of $H(t, a)$ and $H(t, b)$. Using $\deg(h(t, \cdot), \Omega, y(t)) = \deg(H(t, \cdot), \Omega, 0)$ we conclude.

More generally we define the degree for an open and bounded set $\Omega \subseteq \mathbb{R}$. Topological arguments imply that Ω consists of a countable set of distinct connected components

$\Omega_n = (a_n, b_n)$, $n \in \mathbb{N}$ (that is, $\Omega = \dot{\bigcup}_{n=1}^{\infty} \Omega_n$ and $a_n, b_n \notin \Omega$). Obviously, $|a_n - b_n| \to 0$ as $n \to \infty$ because Ω is bounded.

Suppose that $f : \overline{\Omega} \to \mathbb{R}$ is continuous and such that $0 \notin f(\partial\Omega)$. We show that only for a finite number of indices n it holds true that $\deg(f, \Omega_n, 0) \neq 0$: Suppose that n_k is an infinite sequence of indices such that $\deg(f, \Omega_{n_k}, 0) \neq 0$. Since Ω is bounded and since $a_{n_k}, b_{n_k} \in \partial\Omega$, after passing to a subsequence there is $x \in \partial\Omega$ such that $a_{n_k}, b_{n_k} \to x$. Therefore, $f(x) \neq 0$ and $f(a_{n_k})f(b_{n_k}) > 0$ for k sufficiently large. Hence $\deg(f, \Omega_{n_k}, 0) = 0$ for k sufficiently large. Contradiction!

We may now define

$$\deg(f, \Omega, 0) := \sum_{n=1}^{\infty} \deg(f, \Omega_n, 0).$$

As before, for $y \in \mathbb{R} \backslash f(\partial\Omega)$ we define

$$\deg(f, \Omega, y) := \deg(f - y, \Omega, 0).$$

One can show that this degree has similar properties as the degree on an interval. It has image $\mathbb{Z}$, that is, for all $m \in \mathbb{Z}$ there are an open and bounded set $\Omega \subseteq \mathbb{R}$, $f \in C(\overline{\Omega})$ and $y \in \mathbb{R} \backslash f(\partial\Omega)$ such that $\deg(f, \Omega, y) = m$.

1.3 The Degree in Two Dimensions

In two dimensions we identify $\mathbb{R}^2$ with $\mathbb{C}$. For the moment we will define the degree only in the special case of continuously differentiable functions and the open ball with center 0 and radius 1: $\Omega := \{z \in \mathbb{C} \mid |z| < 1\}$, $f \in C^1(\overline{\Omega}, \mathbb{C})$ and $y \in \mathbb{C} \backslash f(\partial\Omega)$. Define the cycle $\gamma \in C^1([0, 1], \mathbb{C})$ by $\gamma(t) := \mathrm{e}^{2\pi \mathrm{i} t}$. Then $\partial\Omega = |\gamma| := \mathcal{R}(\gamma)$. We define

$$\deg(f, \Omega, y) := \frac{1}{2\pi \mathrm{i}} \int_{f \circ \gamma} \frac{1}{z - y} \, \mathrm{d}z. \tag{1.4}$$

The integral is the winding number of the cycle $f \circ \gamma$ with respect to y. With the properties of the winding number one proves that items (i) and (iii) of the previous section hold true, and that also (iv) holds true if h and y are continuously differentiable.

1.2 Example. We define $\varphi_n \in C^1(\overline{\Omega}, \mathbb{C})$ by $\varphi_n(z) := z^n$, where $n \in \mathbb{N}_0$, and we calculate the degree of φ_n with respect to $y = 0$. Then $(\varphi_n \circ \gamma)(t) = \exp(2\pi \mathrm{i} n t)$. Consequently,

$$\deg(\varphi_n, \Omega, 0) = \frac{1}{2\pi \mathrm{i}} \int_{\varphi_n \circ \gamma} \frac{1}{z} \, \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_0^1 \frac{2\pi \mathrm{i} n \exp(2\pi \mathrm{i} n t)}{\exp(2\pi \mathrm{i} n t)} \, \mathrm{d}t = n.$$

2 Construction of the Degree

In this section we will show that there is a unique mapping "deg", the *(mapping) degree*, which assigns integers to triplets consisting of an open bounded subset $\Omega \subseteq \mathbb{R}^N$, a continuous function $f\colon \overline{\Omega} \to \mathbb{R}^N$ and a point $y \in \mathbb{R}^N \setminus f(\partial\Omega)$, and which has the following basic properties:

(D1) $\deg(\mathrm{id}, \Omega, y) = 1$ if $y \in \Omega$,

(D2) $\deg(f, \Omega, y) = \deg(f, \Omega_1, y) + \deg(f, \Omega_2, y)$ if Ω_1, Ω_2 are open subsets of Ω such that $y \notin f(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$,

(D3) $\deg(h(t, \cdot), \Omega, y(t))$ is independent of $t \in J := [0, 1]$ if $h\colon J \times \overline{\Omega} \to \mathbb{R}^N$ is continuous, $y\colon J \to \mathbb{R}^N$ is continuous and $y(t) \notin h(t, \partial\Omega)$ for all $t \in J$.

2.1 Tools

Before we begin we need to establish notation and some topological concepts.

2.1.1 Topological Concepts and Function Spaces

We denote $\mathbb{R}^+ := (0, \infty)$, $\mathbb{R}^- := (-\infty, 0)$, $\mathbb{R}_0^\pm := \mathbb{R}^\pm \cup \{0\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

In this section suppose that X, Y are metric spaces with metrics d_X and d_Y. Often we will omit the index in metrics and norms if the context is clear.

Notation. Suppose that $x \in X$ and $r > 0$. We define

$$
\begin{aligned}
B_r(x) &:= B_r(x; X) := \{\, y \in X \mid d(x, y) < r \,\}, && \textit{open ball},\\
\overline{B}_r(x) &:= \overline{B}_r(x; X) := \{\, y \in X \mid d(x, y) \le r \,\}, && \textit{closed ball},\\
S_r(x) &:= S_r(x; X) := \{\, y \in X \mid d(x, y) = r \,\}, && \textit{sphere}.
\end{aligned}
$$

2.1 Definition. Suppose that $f, g\colon X \to Y$ are continuous. For $J := [0, 1]$ suppose that $h\colon J \times X \to Y$ is continuous and such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$ for all $x \in X$ (i.e., $h(0, \cdot) = f$ and $h(1, \cdot) = g$). Then h is called a *homotopy from f to g*.

2.2 Definition. Suppose that X is a metric space and E a Banach space. We set

$$
\begin{aligned}
C(X, E) &:= \{\, u\colon X \to E \mid u \text{ is continuous} \,\},\\
C_{\mathrm{B}}(X, E) &:= \{\, u\colon X \to E \mid u \text{ is continuous and } u(X) \text{ is bounded} \,\}.
\end{aligned}
$$

Then $C_{\mathrm{B}}(X,E)$, together with the norm

$$\|u\|_{C_{\mathrm{B}}(X,E)} := \|u\|_{C(X,E)} := \|u\|_\infty := \sup_{x\in X}\|u(x)\|_E,$$

is a Banach space. If $E = \mathbb{R}$, then we write $C(X)$ and $C_{\mathrm{B}}(X)$ instead of $C(X,\mathbb{R})$ and $C_{\mathrm{B}}(X,\mathbb{R})$.

2.3 Remark. If X is compact, then $C(X,E) = C_{\mathrm{B}}(X,E)$.

2.4 Definition. Suppose that $M, N \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^N$ is open. We define for $n \in \mathbb{N}_0$

$$C^n(\Omega,\mathbb{R}^M) := \{\, u\colon \Omega \to \mathbb{R}^M \mid u \text{ is } n \text{ times continuously differentiable in } \Omega \,\},$$
$$C^\infty(\Omega,\mathbb{R}^M) := \bigcap_{n=1}^{\infty} C^n(\Omega,\mathbb{R}^M).$$

2.1.2 The Extension Theorem of Tietze-Dugundji

Suppose that X, Y are sets, $A \subseteq X$, $f\colon A \to Y$, $g\colon X \to Y$. We call g an *extension of f to X* if $g(x) = f(x)$ for all $x \in A$.

2.5 Definition. Suppose that X is a set and that $\mathcal{U}$ and $\mathcal{V}$ are coverings of X. If for all $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subseteq U$, then $\mathcal{V}$ is a *refinement* of $\mathcal{U}$.

2.6 Definition. Suppose that X is a topological space. A collection $\mathcal{A}$ of subsets of X is *locally finite* if every $x \in X$ has a neighborhood U such that $U \cap A \neq \varnothing$ only for a finite number of $A \in \mathcal{A}$.

2.7 Definition. A topological space is *paracompact* if it is Hausdorff and if every open covering admits a locally finite refinement.

Without proof we will use the following result of general topology:

2.8 Theorem (Stone)**.** *Every metric space is paracompact.*

2.9 Theorem (Tietze-Dugundji)**.** *Suppose that X is a metric space, E is a normed space, $A \subseteq X$ is closed and not empty, and $f\colon A \to E$ is continuous. Then f has a continuous extension $g\colon X \to E$ such that $g(X) \subseteq \operatorname{conv}(f(A))$.*

Proof. By Theorem 2.8 there are an index set J and open sets $V_j \subseteq X$ such that $\mathcal{V} := \{\, V_j \mid j \in J \,\}$ is a locally finite covering of $X\backslash A$, and such that for all $j \in J$ there is $x_j \in X\backslash A$ with $V_j \subseteq B_{\mathrm{dist}(x_j,A)/2}(x_j)$. We define for $x \in X\backslash A$

$$\varphi_j(x) := \begin{cases} 0 & x \notin V_j \\ \mathrm{dist}(x,\partial V_j) & x \in V_j \end{cases}$$

and

$$\psi_j(x) := \frac{\varphi_j(x)}{\sum_{k\in J}\varphi_k(x)}.$$

Clearly, φ_j is continuous in $X\backslash A$ (since "dist" is Lipschitz continuous). Since $\mathcal{V}$ is locally finite, the sum is well defined and continuous in $X\backslash A$. Since $\mathcal{V}$ covers $X\backslash A$, the sum is positive in $X\backslash A$. This implies that ψ_j is continuous in $X\backslash A$, that $0 \le \psi_j \le 1$, and that $\sum_{j\in J} \psi_j = 1$.

For all $j \in J$ pick $a_j \in A \cap \overline{B}_{2\operatorname{dist}(x_j,A)}(x_j)$. We define

$$g(x) := \begin{cases} f(x) & x \in A \\ \sum_{j\in J} \psi_j(x) f(a_j) & x \in X\backslash A. \end{cases}$$

Clearly, g is continuous in A because it is an extension of f. The sum is finite in a neighborhood of a point $x \in X\backslash A$. It follows that g is continuous in $X\backslash A$. It remains to prove that $g|_{\overline{X\backslash A}}$ is continuous in ∂A, because $X = A \cup \overline{X\backslash A}$ and $A \cap \overline{X\backslash A} = \partial A$.

Let $x^* \in \partial A$ and $\varepsilon > 0$. We fix $\delta > 0$ such that $\|f(a) - f(x^*)\| \le \varepsilon$ if $a \in A \cap \overline{B}_\delta(x^*)$. Suppose that $x \in (X\backslash A) \cap \overline{B}_{\delta/6}(x^*)$. If $j \in J$ satisfies $x \in V_j$, then

$$2d(x, x_j) \le \operatorname{dist}(x_j, A) \le d(x_j, x^*) \le d(x_j, x) + d(x, x^*),$$

that is,

$$d(x, x_j) \le d(x, x^*). \tag{2.1}$$

We also obtain that

$$d(x_j, a_j) \le 2\operatorname{dist}(x_j, A) \le 2d(x_j, x^*) \le 2(d(x_j, x) + d(x, x^*)) \le 4d(x, x^*). \tag{2.2}$$

Equations (2.1) and (2.2) imply that

$$d(x^*, a_j) \le d(x^*, x) + d(x, x_j) + d(x_j, a_j) \le 6d(x^*, x) \le \delta.$$

Hence $\|f(x^*) - f(a_j)\| \le \varepsilon$ for all $j \in J$ such that $x \in V_j$. This implies that

$$\|g(x) - f(x^*)\| = \left\| \sum_{j\in J} \psi_j(x)(f(a_j) - f(x^*)) \right\| \le \sum_{j\in J} \psi_j \varepsilon = \varepsilon.$$

Since $x^* \in \partial A$ and ε were arbitrary, this finishes the continuity proof for g.

It is obvious that $g(x)$ is a convex combination of elements of $f(A)$ for all $x \in X$. □

2.1.3 Sard's Lemma

2.10 Definition. If $\Omega \subseteq \mathbb{R}^N$ is open and $f\colon \Omega \to \mathbb{R}^N$ continuously differentiable then we call $J_f(x) := \det \mathrm{D}f(x)$ the *Jacobean of f in x*. If $J_f(x) = 0$ then x is a *critical point of f*. If $J_f(x) \ne 0$ then x is a *regular point of f*. We write $K_f(\Omega) := \{\, x \in \Omega \mid J_f(x) = 0 \,\}$, and we write K_f if from context it is clear which is Ω. We say that $y \in \mathbb{R}^N$ is a *regular value* of $f\colon \Omega \to \mathbb{R}^N$ if $f^{-1}(y) \cap K_f(\Omega) = \varnothing$, and a *singular value* otherwise.

2.11 Lemma. *Suppose that $\Omega \subseteq \mathbb{R}^N$ is open and bounded, $f \in C(\overline{\Omega}, \mathbb{R}^N) \cap C^1(\Omega, \mathbb{R}^N)$, and $y \in \mathbb{R}^N\backslash f(\partial\Omega)$. If y is a regular value of f then $f^{-1}(y)$ is finite.*

Proof. $K := f^{-1}(y) \subseteq \Omega$ is compact because f is continuous and Ω bounded. Arguing by contradiction, suppose that K is infinite. Then there is a sequence $(x_n) \subseteq K$ such that $x_m \neq x_n$ if $m \neq n$. Since K is compact we may assume that there is $x \in K$ such that $x_n \to x$ as $n \to \infty$. Since y is regular, the inverse function theorem implies that there are open neighborhoods U of x and V of y such that $f|U$ is a bijection between U and V, that is, $U \cap K = \{x\}$. But this contradicts $x_n \to x$ in K! □

2.12 Definition. If $a_k \leq b_k$ for $k = 1, 2, \dots, N$ then $K := \prod_{k=1}^{N}[a_k, b_k]$ is a *brick (rectangular parallelepiped) of dimension N with volume* $|K| := \prod_{k=1}^{N}(b_k - a_k)$.

More generally, for a Lebesgue measurable subset $A \subseteq \mathbb{R}^N$ we denote by $|A|$ its Lebesgue measure.

Recall that for normed spaces E, F we denote by $\mathcal{L}(E, F)$ the vector space of bounded linear operators from E to F, together with norm

$$\|L\|_{\mathcal{L}(E,F)} := \sup_{x \in \overline{B}_1 E} |Lx|.$$

If F is Banach then $\mathcal{L}(E, F)$ is also a Banach space.

2.13 Proposition (Sard's Lemma)**.** *Suppose that $\Omega \subseteq \mathbb{R}^N$ is open and $f \in C^1(\Omega, \mathbb{R}^N)$. Then $|f(K_f)| = 0$.*

Proof. Suppose that

$$\mathcal{Q} := \left\{ \prod_{k=1}^{N} \left[x^k - \frac{1}{2n}, x^k + \frac{1}{2n} \right] \,\middle|\, n \in \mathbb{N},\ nx \in \mathbb{Z}^N \right\}.$$

Then $\mathcal{Q}$ is a countable family of closed cubes covering $\mathbb{R}^N$. We set

$$\mathcal{R} := \{\, Q \in \mathcal{Q} \mid Q \subseteq \Omega \,\}.$$

It is clear that $\mathcal{R}$ is countable and covers Ω. Therefore, it is sufficient prove that $|f(K_f \cap Q)| = 0$ for all $Q \in \mathcal{R}$.

Suppose that $Q \in \mathcal{R}$ is a cube of side length $r > 0$. Since f is continuously differentiable in a neighborhood of Q we may define $C_1 := \max_{x \in Q} \|\mathrm{D}f(x)\|$. Note that $f(K_f \cap Q)$ is compact and hence measurable. Suppose that $\varepsilon \in (0, 1]$. It is sufficient to show that $|f(K_f \cap Q)| \leq C_2 \varepsilon$, where C_2 does not depend on ε. Since $\mathrm{D}f$ is continuous and Q is compact, $\mathrm{D}f$ is uniformly continuous on Q. There is $m(\varepsilon) \in \mathbb{N}$ such that for $\delta(\varepsilon) := \sqrt{N}\, r / m(\varepsilon)$ it holds true that:

$$\|\mathrm{D}f(x) - \mathrm{D}f(y)\| \leq \varepsilon \qquad \text{for all } x, y \in Q,\ |x - y| \leq \delta(\varepsilon).$$

Hence

$$
\begin{aligned}
(2.3)\quad |f(x) - f(y) - \mathrm{D}f(y)[x-y]| \\
&= \left| \int_0^1 \mathrm{D}f((1-t)y + tx)[x-y]\,\mathrm{d}t - \mathrm{D}f(y)[x-y] \right| \\
&= \left| \int_0^1 \big(\mathrm{D}f((1-t)y+tx) - \mathrm{D}f(y)\big)[x-y]\,\mathrm{d}t \right| \\
&\le \int_0^1 \|\mathrm{D}f((1-t)y+tx) - \mathrm{D}f(y)\|\,|x-y|\,\mathrm{d}t \\
&\le \varepsilon\delta(\varepsilon)
\end{aligned}
$$

for all $x, y \in Q$, $|x-y| \le \delta(\varepsilon)$. We decompose Q into $n(\varepsilon)$ cubes Q_k, $k = 1, 2, \dots, n(\varepsilon)$, with sides parallel to the canonical base, with pairwise disjoint interiors and each with diameter $\delta(\varepsilon)$. The side length of Q_k is $\delta(\varepsilon)/\sqrt{N} = r/m(\varepsilon)$. Therefore, $n(\varepsilon) = m(\varepsilon)^N$.

We prove that there is $C_3 \ge 0$, independent of ε and k, such that

$$(2.4)\qquad f(Q_k) \text{ is measurable and } |f(Q_k)| \le C_3\varepsilon\delta(\varepsilon)^N \text{ if } K_f \cap Q_k \ne \varnothing.$$

Suppose that $x^* \in K_f \cap Q_k$. We define $L := \mathrm{D}f(x^*)$, $\widetilde{Q_k} := Q_k - x^*$ and $g\colon \widetilde{Q_k} \to \mathbb{R}^N$ by

$$g(x) := f(x^* + x) - f(x^*).$$

Then (2.3) implies that

$$(2.5)\qquad |g(x) - Lx| \le \varepsilon\delta(\varepsilon) \qquad \text{for all } x \in \widetilde{Q_k}.$$

Since $J_f(x^*) = \det(L) = 0$, the image of L is contained in a subspace A of $\mathbb{R}^N$ of dimension $N-1$. Pick $b_1 \in S_1\mathbb{R}^N \cap A^\perp$ and complete $\{b_1\}$ to an orthonormal base $\{\, b_1, b_2, \dots, b_N \,\}$ of $\mathbb{R}^N$. Using (2.5), $(Lx)\cdot b_1 = 0$ and $|b_1| = 1$ we calculate for $x \in \widetilde{Q_k}$:

$$|g(x)\cdot b_1| = |(g(x) - Lx)\cdot b_1| \le |g(x) - Lx| \le \varepsilon\delta(\varepsilon).$$

For $i = 2, 3, \dots, N$ we obtain for $x \in \widetilde{Q_k}$ that

$$|g(x)\cdot b_i| = |(g(x) - Lx)\cdot b_i| + |(Lx)\cdot b_i| \le |g(x) - Lx| + \|L\|\,|x| \le (\varepsilon + \|L\|)\delta(\varepsilon)$$

because $\widetilde{Q_k} \subseteq \overline{B}_{\delta(\varepsilon)}\mathbb{R}^N$. This implies that $g(\widetilde{Q_k})$ is contained in a rotated brick with sides that are parallel to the b_i, with volume $2^N(1+\|L\|)^{N-1}\varepsilon\delta(\varepsilon)^N$. Setting $C_3 := 2^N(1+C_1)^{N-1}$ and using $f(Q_k) = f(x^*) + g(\widetilde{Q_k})$, we obtain (2.4), noting that $f(Q_k)$ is compact and therefore measurable.

There are no more than $n(\varepsilon) = m(\varepsilon)^N$ cubes Q_k that satisfy $K_f \cap Q_k \ne \varnothing$. Since $\{Q_k\}_{k=1}^{n(\varepsilon)}$ is a covering of Q,

$$f(K_f \cap Q) \subseteq \bigcup_{K_f \cap Q_k \ne \varnothing} f(Q_k).$$

From (2.4) we conclude that

$$|f(K_f \cap Q)| \le \sum_{K_f \cap Q_k \ne \varnothing} |f(Q_k)| \le m(\varepsilon)^N C_3\varepsilon\delta(\varepsilon)^N = C_3 N^{N/2} r^N \varepsilon = C_2\varepsilon,$$

where $C_2 := C_3 N^{N/2} r^N$ is independent of ε. □

2.1.4 Smoothing

2.14 Definition. Suppose that $\Omega \subseteq \mathbb{R}^N$ is open. If $f \in C(\Omega, \mathbb{R}^M)$, then we call

$$\operatorname{supp}(f) := \overline{\{ x \in \mathbb{R}^N \mid f(x) \neq 0 \}} \cap \Omega$$

the *support* of f. We define for $n \in \mathbb{N}_0$

$$C_c^n(\Omega) := \{ u \in C^n(\Omega) \mid \operatorname{supp}(u) \text{ is compact} \}$$
$$C_c^\infty(\Omega) := \bigcap_{n=1}^{\infty} C_c^n(\Omega).$$

2.15 Definition. Fixing $N \in \mathbb{N}$ we define $\eta \in C_c^\infty(\mathbb{R}^N)$ by

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) & |x| < 1 \\ 0 & |x| \geq 1, \end{cases}$$

where $C > 0$ is such that $\int_{\mathbb{R}^N} \eta \, \mathrm{d}x = 1$. For $\delta > 0$ we define

$$\eta_\delta(x) := \frac{1}{\delta^N} \eta\left(\frac{x}{\delta}\right).$$

It follows that $\eta_\delta \in C_c^\infty(\mathbb{R}^N)$, that

$$\operatorname{supp}(\eta_\delta) = \overline{B}_\delta(0), \tag{2.6}$$

and that

$$\int_{\mathbb{R}^N} \eta_\delta \, \mathrm{d}x = 1 \tag{2.7}$$

for all $\delta > 0$. The functions η_δ form a family of *mollifiers*.

2.16 Proposition. *Suppose that $u \in C(\mathbb{R}^N)$. For $\delta > 0$ we define the smoothing $u_\delta \colon \mathbb{R}^N \to \mathbb{R}$ of u by the convolution of u and η_δ:*

$$u_\delta(x) := (\eta_\delta * u)(x) := \int_{\mathbb{R}^N} \eta_\delta(x-y) u(y) \, \mathrm{d}y = \int_{\overline{B}_\delta(x)} \eta_\delta(x-y) u(y) \, \mathrm{d}y. \tag{2.8}$$

Then u_δ has the following properties:

(i) $u_\delta \in C^\infty(\mathbb{R}^N)$,

(ii) u_δ converges uniformly to u in compact subsets of $\mathbb{R}^N$ as $\delta \to 0$,

(iii) if $\Omega \subseteq \mathbb{R}^N$ is open and if u is continuously differentiable in Ω then $\partial_i u_\delta$ converges uniformly to $\partial_i u$ in compact subsets of Ω as $\delta \to 0$, for all $i = 1, 2, \ldots, N$.

2.2 Uniqueness

Our goal here is to deduce, step by step, the necessary form of the mapping degree from its proposed properties.

2.2.1 Reduction to Linear Operators

2.17 Proposition. *A degree with properties* (D1)–(D3) *is determined by its values on triplets* (g, Ω, z)*, where* $\Omega \subseteq \mathbb{R}^N$ *is open and bounded,* $g \in C(\overline{\Omega}, \mathbb{R}^N) \cap C^\infty(\Omega, \mathbb{R}^N)$ *and* $z \in \mathbb{R}^N \backslash g(\partial\Omega)$ *is a regular value of* g*.*

Proof. Suppose that $\Omega \subseteq \mathbb{R}^N$ is open and bounded, $f \in C(\overline{\Omega}, \mathbb{R}^N)$ and $y \in \mathbb{R}^N \backslash f(\partial\Omega)$. We set $\delta := \operatorname{dist}(y, f(\partial\Omega)) > 0$. By Theorem 2.9 we may assume that $f \in C(\mathbb{R}^N, \mathbb{R}^N)$. With Proposition 2.16 and its items (i) and (ii), applied to each of the components of f, we construct $g \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ such that $\sup_{x\in\overline{\Omega}}|f(x) - g(x)| \leq \delta/3$. Using Sard's Lemma (Proposition 2.13) we pick a regular value $z \in \overline{B}_{\delta/3}(y)$ of g. Moreover, we define the homotopy $h(t, x) := (1-t)f(x) + tg(x)$ and the map $w(t) := (1-t)y + tz$. If $t \in [0, 1]$ and $x \in \partial\Omega$, then

$$\begin{aligned} |h(t, x) - w(t)| &= |f(x) - y - t(f(x) - g(x) + z - y)| \\ &\geq |f(x) - y| - t(|f(x) - g(x)| + |y - z|) \geq \delta - t(\delta/3 + \delta/3) \geq \delta/3. \end{aligned}$$

Hence $w(t) \notin h(t, \partial\Omega)$ for all $t \in [0, 1]$ and (D3) imply that

$$\deg(f, \Omega, y) = \deg(g, \Omega, z) \tag{2.9}$$

if "deg" is a degree that satisfies (D1)–(D3). □

2.18 Proposition. *A degree with the properties* (D1)–(D3) *is determined by its values on triplets* $(L, B_1, 0)$*, where* $L \in \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)$ *is an isomorphism.*

Proof. Suppose that $\Omega \subseteq \mathbb{R}^N$ is open and bounded, $f \in C(\overline{\Omega}, \mathbb{R}^N)$ and $y \in \mathbb{R}^N \backslash f(\partial\Omega)$.

First suppose that $f^{-1}(y) = \varnothing$. By definition, $\varnothing$ is an open subset of Ω. Using $\Omega_1 := \Omega$ and $\Omega_2 := \varnothing$ (D2) implies that $\deg(f, \varnothing, y) = 0$. Using $\Omega_1 := \varnothing$, $\Omega_2 := \varnothing$ and $f^{-1}(y) = \varnothing$ (D2) implies that $\deg(f, \Omega, y) = 0$.

If $f^{-1}(y) \neq \varnothing$ we may assume by Proposition 2.17 that $f \in C(\overline{\Omega}, \mathbb{R}^N) \cap C^\infty(\Omega, \mathbb{R}^N)$ and $y \notin f(K_f)$. Lemma 2.11 implies that there are $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in \Omega$ such that $f^{-1}(y) = \{ x_1, x_2, \ldots, x_n \}$. There is $r > 0$ (which can be selected arbitrarily small) such that the balls $B_r(x_k)$ are disjoint. Property (D2) implies that

$$\deg(f, \Omega, y) = \sum_{k=1}^{n} \deg(f, B_r(x_k), y). \tag{2.10}$$

Hence $\deg(f, \Omega, y)$ is determined by the values of $\deg(f, B_r(x_k), y)$ $(k = 1, 2, \ldots, n)$.

To calculate $\deg(f, B_r(x_k), y)$ fix k and define $L := \mathrm{D}f(x_k)$. Since L is an isomorphism we have for $x \in \mathbb{R}^N$ that $|x| = |L^{-1}Lx| \leq \|L^{-1}\| \, |Lx|$, that is,

$$|Lx| \geq \frac{|x|}{\|L^{-1}\|} \qquad \text{for all } x \in \mathbb{R}^N. \tag{2.11}$$

Moreover, $f(x_k) = y$ implies that

$$\frac{|f(x) - y - L[x - x_k]|}{|x - x_k|} \to 0 \qquad \text{as } x \to x_k.$$

Fix r small enough such that

$$|f(x) - y - L[x - x_k]| \leq \frac{|x - x_k|}{2\|L^{-1}\|} \qquad \text{for all } x \in \overline{B}_r(x_k). \tag{2.12}$$

We define the homotopy $h(t, x) := (1-t)f(x) + tL[x - x_k]$ and the map $w(t) := (1-t)y$. With (2.11) and (2.12) we calculate for $t \in [0, 1]$ and $x \in S_r(x_k)$:

$$\begin{aligned} |h(t, x) - w(t)| &= |L[x - x_k] + (1-t)(f(x) - y - L[x - x_k])| \\ &\geq |L[x - x_k]| - (1-t)|f(x) - y - L[x - x_k]| \geq \frac{|x - x_k|}{2\|L^{-1}\|} = \frac{r}{2\|L^{-1}\|} > 0. \end{aligned}$$

It follows that $w(t) \notin h(t, \partial B_r(x_k))$ for all $t \in [0, 1]$, and (D3) implies that

$$\deg(f, B_r(x_k), y) = \deg(L[\,\cdot\, - x_k], B_r(x_k), 0). \tag{2.13}$$

Pick $R > 0$ sufficiently large such that $B_r(x_k) \subseteq B_R(0)$. Since L is an isomorphism, $L[x - x_k] = 0$ only for $x = x_k$. Hence (D2) implies (using $\Omega_1 := B_r(x_k)$ and $\Omega_2 := \varnothing$) that

$$\deg(L[\,\cdot\, - x_k], B_r(x_k), 0) = \deg(L[\,\cdot\, - x_k], B_R(0), 0). \tag{2.14}$$

The homotopy $h(t, x) := (1-t)L[x - x_k] + tLx$ satisfies for $t \in [0, 1]$ and $x \in S_R(0)$ that

$$|h(t, x)| = |L[x - (1-t)x_k]| \geq \frac{|x - (1-t)x_k|}{\|L^{-1}\|} \geq \frac{|x| - |x_k|}{\|L^{-1}\|} \geq \frac{R - (R - r)}{\|L^{-1}\|} > 0.$$

Hence (D3) implies that

$$\deg(L[\,\cdot\, - x_k], B_R(0), 0) = \deg(L, B_R(0), 0). \tag{2.15}$$

Another application of (D2) proves that

$$\deg(L, B_R(0), 0) = \deg(L, B_1(0), 0). \tag{2.16}$$

Combining (2.10) with (2.13)–(2.16) we conclude that

$$\deg(f, \Omega, y) = \sum_{k=1}^{n} \deg(\mathrm{D}f(x_k), B_1, 0). \tag{2.17}$$

□

2.2.2 The Degree of a Linear Operator

2.19 Lemma. *Suppose that E is a normed space of finite, positive and even dimension. Then there is a homotopy h of $-\mathrm{id}_E$ to id_E such that*

$$h(t,x) = 0 \qquad \textit{implies that } x = 0.$$

Proof. Suppose that $\dim(E) = 2k$ with $k \in \mathbb{N}$. For $t \in [0,1]$ we define a linear operator $A(t) \in \mathcal{L}(E)$, represented by a matrix with respect to a fixed base of E. First we set

$$B(t) := \begin{pmatrix} -\cos(\pi t) & \sin(\pi t) \\ -\sin(\pi t) & -\cos(\pi t) \end{pmatrix},$$

and then

$$A(t) := \underbrace{\begin{pmatrix} B(t) & 0 & 0 & \cdots & 0 \\ 0 & B(t) & 0 & \cdots & 0 \\ 0 & 0 & B(t) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & B(t) \end{pmatrix}}_{k}.$$

Is clear that $\det(A(t)) = 1$ and that hence $A(t)$ is an isomorphism for all t. Moreover, with respect to the norm in $\mathcal{L}(E)$, $A(t)$ depends continuously on t because the coordinates of $A(t)$ are continuous. If $t_n \to t$ and $x_n \to x$ then

$$\begin{aligned} \|A(t_n)x_n - A(t)x\|_E &\le \|A(t_n)x_n - A(t)x_n\|_E + \|A(t)x_n - A(t)x\|_E \\ &\le \|x_n\|_E \|A(t_n) - A(t)\|_{\mathcal{L}(E)} + \|A(t)\|_{\mathcal{L}(E)} \|x_n - x\|_E \to 0 \end{aligned}$$

because $A \in C([0,1], \mathcal{L}(E))$ and $\|x_n\|_E \to \|x\|_E$. Hence $h(t,x) := A(t)x$ is continuous. Is clear that $h(0,x) = -x$ and $h(1,x) = x$. Since $A(t)$ is an isomorphism, $A(t)x = 0$ implies that $x = 0$. □

2.20 Remark. By (D1) and (D3), Lemma 2.19 implies that $\deg(-\mathrm{id}, B_1\mathbb{R}^{2k}, 0) = 1$ for $k \in \mathbb{N}$.

2.21 Lemma. *Suppose that "*deg*" is a degree with* (D1)–(D3) *in dimension* 1*. Suppose moreover that $\Omega \subseteq \mathbb{R}$ is open, bounded, and such that $0 \in \Omega$. Then* $\deg(-\mathrm{id}, \Omega, 0) = -1$.

Proof. By (D2) we may assume that $\Omega = B_1$. We define $f \in C(\overline{B}_2(1))$ by $f(x) := |x-1| - 1$. Since f and 1 have the same values on $\partial B_2(1)$, it follows by Exercise Sheet 2, N° 1, that

$$\deg(f, B_2(1), 0) = \deg(1, B_2(1), 0) = 0.$$

Here we used the argument of the second paragraph of the proof of Proposition 2.18, i.e. that the function 1 has no zero. It follows that

$$\deg(f, B_1(2), 0) = \deg(\mathrm{id} - 2, B_1(2), 0) \overset{\text{(D3)}}{=} \deg(\mathrm{id}, B_1(2), 2) \overset{\text{(D1)}}{=} 1.$$

hence (D2) and $0 \notin f(\overline{B}_2(1) \backslash (B_1(0) \cup B_1(2)))$ imply that

$$\begin{aligned} \deg(-\mathrm{id}, B_1(0), 0) &= \deg(f, B_1(0), 0) \\ &= \deg(f, B_2(1), 0) - \deg(f, B_1(2), 0) = 0 - 1 = -1. \end{aligned}$$

□

2.22 Proposition. *Suppose that A is a linear isomorphism in $\mathbb{R}^N$ and "deg" a degree that satisfies* (D1)–(D3). *Then* $\deg(A, B_1, 0) = \operatorname{sgn}(\det(A))$.

Proof. From the Jordan normal form of a matrix we deduce the existence of invariant subspaces F, G for A such that $\mathbb{R}^N = F \oplus G$, $\operatorname{sgn}(\det(A)) = (-1)^{\dim(F)}$, $A|_F$ has only negative eigenvalues and $A|_G$ has only eigenvalues in $\mathbb{C} \backslash (-\infty, 0]$. Denote by P_F, P_G the projections corresponding to the splitting $F \oplus G$. Since F, G are invariant, $P_F A P_F = A P_F$, $P_G A P_G = A P_G$ and $P_F A P_G = P_G A P_F = 0$. Hence $P_F A = P_F A(P_F + P_G) = P_F A P_F = A P_F$ and similarly $P_G A = A P_G$.

We define

$$h(t, x) := (1 - t)Ax + t(-P_F + P_G)x.$$

If $t \in [0, 1]$ and $x \in \mathbb{R}^N$ satisfy $h(t, x) = 0$, then

$$0 = P_F h(t, x) = (1 - t)P_F Ax - tP_F x = (1 - t)AP_F x - tP_F x.$$

For $t = 1$ we obtain $P_F x = 0$. For $t < 1$ it follows that

$$AP_F x = \frac{t}{1 - t} P_F x.$$

Hence $P_F x = 0$ because $A|_F$ has only negative eigenvalues. On the other hand we have

$$0 = P_G h(t, x) = (1 - t)AP_G x + tP_G x.$$

For $t = 1$ we obtain $P_G x = 0$. For $t < 1$ it follows that

$$AP_G x = -\frac{t}{1 - t} P_G x.$$

Therefore, $P_G x = 0$ because $A|_G$ only has eigenvalues in $\mathbb{C} \backslash (-\infty, 0]$. Summing up we proved that $x = P_F x + P_G x = 0$ if $h(t, x) = 0$ and $t \in [0, 1]$. Using (D3) we obtain that

$$\deg(A, B_1, 0) = \deg(-P_F + P_G, B_1, 0). \tag{2.18}$$

Case 1: If $\dim(F) = 0$ this implies that

$$\deg(A, B_1, 0) = \deg(\mathrm{id}, B_1, 0) = 1 = (-1)^{\dim(F)} = \operatorname{sgn}(\det(A)).$$

Case 2: For $\dim(F) > 0$ we distinguish two cases: If $\dim(F)$ is even then we define $D := \{0\}$ and $E := F$. If $\dim(F)$ is odd then we pick D, a subspace of F of dimension

1, and E, a subspace of F with even dimension such that $F = D \oplus E$. We define $Q_D, Q_E \in \mathcal{L}(F)$ as the projections with respect to this decomposition of F.

Using Lemma 2.19 we find $h_1 \in C([0,1]\times E, E)$ such that $h_1(0,\cdot) = -\mathrm{id}_E$, $h_1(1,\cdot) = \mathrm{id}_E$, and such that $h_1(t,x) = 0$ implies $x = 0$. We define

$$h_2(t,x) := -Q_D P_F x + h_1(t, Q_E P_F x) + P_G x.$$

Suppose that $t \in [0,1]$ and $x \in \mathbb{R}^N$ are such that $h_2(t,x) = 0$. We obtain $0 = P_G h_2(t,x) = P_G x$, $0 = -Q_D P_F h_2(t,x) = Q_D P_F x$ and $0 = Q_E P_F h_2(t,x) = h_1(t, Q_E P_F x)$. The properties of h_1 imply that $Q_E P_F x = 0$. We have therefore proved that

$$x = P_F x + P_G x = Q_D P_F x + Q_E P_F x + P_G x = 0 \qquad \text{if } h_2(t,x) = 0,\ t \in [0,1]. \tag{2.19}$$

We set $H := E \oplus G$. It follows that $\mathbb{R}^N = D \oplus H$, and that $P_D := Q_D P_F$ and $P_H := Q_E P_F + P_G$ are the projections corresponding to this decomposition of $\mathbb{R}^N$. The definition of h_2 and (2.19) imply by (D3) that

$$\deg(-P_F + P_G, B_1(0), 0) = \deg(-P_D + P_H, B_1, 0). \tag{2.20}$$

Case 2a): If $\dim(F)$ is even then $P_D = 0$ and $P_H = \mathrm{id}$. It follows by (2.18), (2.20) and (D1) that

$$\deg(A, B_1, 0) = 1 = (-1)^{\dim(F)} = \mathrm{sgn}(\det(A)).$$

Case 2b): If $\dim(F)$ is odd then by (2.18), (2.20) and (D1) it only remains to prove that

$$\deg(-P_D + P_H, B_1\mathbb{R}^N, 0) = -1, \tag{2.21}$$

because $-1 = (-1)^{\dim(F)} = \mathrm{sgn}(\det(A))$. Let us recall that $\dim(D) = 1$. Suppose that $L\colon \mathbb{R} \to D$ a linear isomorphism. We define another degree $\deg_1$ in dimension 1: Suppose that $\Omega \subseteq \mathbb{R}$ is open and bounded, $f \in C(\overline{\Omega})$ and $y \in \mathbb{R}\backslash f(\partial\Omega)$. We define $\Omega_* := L(\Omega) + B_1 H$, $f_* \in C(\overline{\Omega_*}, \mathbb{R}^N)$ by

$$f_*(x) := Lf(L^{-1}P_D x) + P_H x$$

and $y_* := Ly$ and prove that $y_* \notin f_*(\partial\Omega_*)$: arguing by contradiction, suppose that there is $x \in \partial\Omega_*$ such that $f_*(x) = Lf(L^{-1}P_D x) + P_H x = y_*$. It is clear that $P_H x = 0$ because $y_* \in D$. Hence $P_D x = x \in \partial\Omega_*$, that is, $L^{-1}P_D x \in \partial\Omega$, because clearly $\partial\Omega_* \cap D = \partial(L(\Omega)) = L(\partial\Omega)$. Moreover,

$$y = L^{-1}y_* = f(L^{-1}P_D x),$$

contradicting $y \notin f(\partial\Omega)$. Hence we can define

$$\deg_1(f, \Omega, y) := \deg(f_*, \Omega_*, y_*).$$

We prove the properties (D1)–(D3) for $\deg_1$:

(D1): Suppose that $y \in \Omega$. It follows that $y_* \in \Omega_*$ and that

$$\deg_1(\mathrm{id}, \Omega, y) = \deg(P_D + P_H, \Omega_*, y_*) = \deg(\mathrm{id}, \Omega_*, y_*) = 1.$$

(D2): Suppose that $\Omega_1, \Omega_2 \subseteq \Omega$ are open and $y \notin f(\overline{\Omega} \backslash (\Omega_1 \cup \Omega_2))$. By contradiction, suppose that there is $x \in \overline{\Omega_*} \backslash (\Omega_{1*} \cup \Omega_{2*})$ such that $f_*(x) = y_*$. As before, $P_H x = 0$ and $L^{-1} P_D x \in \overline{\Omega} \backslash (\Omega_1 \cup \Omega_2)$. Moreover, $f(L^{-1} P_D x) = y$, a contradiction. Hence $y_* \notin f_*(\overline{\Omega_*} \backslash (\Omega_{1*} \cup \Omega_{2*}))$ and we obtain that

$$\begin{aligned} \deg_1(f, \Omega, y) &= \deg(f_*, \Omega_*, y_*) \\ &= \deg(f_*, \Omega_{1*}, y_*) + \deg(f_*, \Omega_{2*}, y_*) \\ &= \deg_1(f, \Omega_1, y) + \deg_1(f, \Omega_2, y). \end{aligned}$$

(D3): Suppose that $h \in C([0,1] \times \overline{\Omega})$ and $y \in C([0,1])$ are such that $y(t) \notin h(t, \partial\Omega)$ for all $t \in [0,1]$. We define $h_*(t,x) := Lh(t, L^{-1} P_D x) + P_H x$ for $x \in \mathbb{R}^N$ and $y_*(t) := Ly(t)$. By contradiction, suppose that there are $t \in [0,1]$ and $x \in \partial\Omega_*$ such that $h_*(t,x) = y_*(t)$. Once again this implies that $L^{-1} P_D x \in \partial\Omega$ and $h(t, L^{-1} P_D x) = y(t)$, a contradiction. Hence $y_*(t) \notin h_*(t, \partial\Omega_*)$ for all $t \in [0,1]$ and we obtain that

$$\deg_1(h(t,\cdot), \Omega, y) = \deg(h_*(t,\cdot), \Omega_*, y_*(t))$$

is independent of t.

We remark that $-P_D + P_H$ is an isomorphism and that $L \circ (-\mathrm{id}) \circ L^{-1} \circ P_D = -P_D$. Define $\Omega := L^{-1}(B_1 D)$ and observe that $0 \in \Omega$. Using (D2) and Lemma 2.21, we calculate

$$\deg(-P_D + P_H, B_1\mathbb{R}^N, 0) = \deg(-P_D + P_H, B_1 D + B_1 H, 0) = \deg_1(-\mathrm{id}, \Omega, 0) = -1.$$

This proves (2.21) and finishes the proof. $\square$

Summing up, Proposition 2.17, Proposition 2.18 and Proposition 2.22 and its proofs imply

2.23 Theorem. *Suppose that "*deg*" is a degree that satisfies* (D1)–(D3)*. Suppose that $\Omega \subseteq \mathbb{R}^N$ is open and bounded, $f \in C(\overline{\Omega}, \mathbb{R}^N)$ and $y \in \mathbb{R}^N \backslash f(\partial\Omega)$. We set $\rho := \mathrm{dist}(y, f(\partial\Omega))$. Then there are $g \in C(\overline{\Omega}, \mathbb{R}^N) \cap C^\infty(\Omega, \mathbb{R}^N)$ and a regular value $z \in \mathbb{R}^N \backslash g(\partial\Omega)$ of g such that $\|f - g\|_{C(\overline{\Omega}, \mathbb{R}^N)} \leq \rho/3$ and $|y - z| \leq \rho/3$. For all those g and z it holds true that $g^{-1}(z)$ is a finite set and that*

$$\deg(f, \Omega, y) = \sum_{x \in g^{-1}(z)} \operatorname{sgn} J_g(x). \tag{2.22}$$

2.3 Existence

2.3.1 Regular Values and Functions in C^2

2.24 Definition. Suppose that $\Omega \subseteq \mathbb{R}^N$ is open and bounded, $f \in C^1(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$ and $y \in \mathbb{R}^N \backslash f(\partial\Omega \cup K_f)$. We define

$$\deg(f, \Omega, y) := \sum_{x \in f^{-1}(y)} \operatorname{sgn} J_f(x).$$

2.25 Remarks. (a) Lemma 2.11 implies that the sum mentioned above is finite.

(b) If $f^{-1}(y) = \varnothing$, then $\deg(f, \Omega, y) = 0$ because the sum is empty.

We are interested in extending Definition 2.24 to singular values y of f. Suppose that $\rho := \operatorname{dist}(y, f(\partial\Omega)) > 0$. Since $f(K_f)$ has zero measure in $\mathbb{R}^N$ there are regular values in the ball $B_\rho(y)$. It is natural to define the degree $\deg(f, \Omega, y)$ by $\deg(f, \Omega, y_1)$ if $y_1 \in B_\rho(y)$ is a regular value of f. But one has that prove that this definition does not depend on the selection of y_1, and for this we need some tools.

2.26 Lemma. *Suppose that Ω, f and y are as in Definition 2.24. For $\delta > 0$ suppose that η_δ is given by Definition 2.15. Then there is $\delta_0 > 0$ such that*

$$\deg(f, \Omega, y) = \int_\Omega \eta_\delta(f(x) - y) J_f(x)\,\mathrm{d}x \qquad \text{for all } \delta \in (0, \delta_0]. \tag{2.23}$$

Proof. If $f^{-1}(y) = \varnothing$ then pick $0 < \delta_0 < \operatorname{dist}(y, f(\overline{\Omega}))$. It follows that $\eta_\delta(f(x) - y) = 0$ for all $x \in \Omega$ and $\delta \in (0, \delta_0]$ because $\operatorname{supp}(\eta_\delta) = \overline{B}_\delta$. Therefore, (2.23) holds true.

Suppose that $f^{-1}(y) = \{ x_1, x_2, \dots, x_n \}$. We find $r_1 > 0$ such that the closed balls $\overline{B}_{r_1}(x_k) \subseteq \Omega$ are mutually disjoint and each of the restrictions of f to $B_{r_1}(x_k)$ is a diffeomorphism with image V_k, an open neighborhood of y. It follows from the continuity of J_f that

$$\operatorname{sgn} J_f(x) = \operatorname{sgn} J_f(x_k) \qquad \text{for all } x \in B_{r_1}(x_k). \tag{2.24}$$

There is $r_2 > 0$ such that $\overline{B}_{r_2}(y) \subseteq \bigcap_{k=1}^n V_k$ and $f^{-1}(B_{r_2}(y)) \subset \bigcup_{k=1}^n B_{r_1}(x_k)$. We define the open sets $W_k := f^{-1}(B_{r_2}(y)) \cap B_{r_1}(x_k)$. Then $\operatorname{dist}(y, f(\overline{\Omega} \setminus \bigcup_{k=1}^n W_k)) \geq r_2$. If $\delta \leq \delta_0 := r_2$ and $x \in \overline{\Omega} \setminus \bigcup_{k=1}^n W_k$ then $f(x) - y \notin B_\delta$ and $\eta_\delta(f(x) - y) = 0$. Hence (2.24) implies that

$$\int_\Omega \eta_\delta(f(x) - y) J_f(x)\,\mathrm{d}x = \sum_{k=1}^n \operatorname{sgn} J_f(x_k) \int_{W_k} \eta_\delta(f(x) - y) |J_f(x)|\,\mathrm{d}x. \tag{2.25}$$

We observe that $J_{f-y} = J_f$. Moreover, f is a diffeomorphism of W_k and $B_{r_2}(y)$, and therefore $f - y$ is a diffeomorphism of W_k and B_{r_2}. It follows by a change of variables and from $B_\delta \subseteq B_{r_2}$ that

$$\int_{W_k} \eta_\delta(f(x) - y) |J_f(x)|\,\mathrm{d}x = \int_{B_{r_2}} \eta_\delta(z)\,\mathrm{d}z = 1.$$

Using this, (2.25) implies (2.23). □

2.27 Definition. Suppose that $A = ((a_{ij})) \in \mathbb{R}^{N \times N}$ is a matrix. By definition, the *cofactor* $\det_{ij}(A)$ is $(-1)^{i+j}$ times the determinant of A after taking away the i-th row and the j-th column:

$$\det_{ij}(A) := (-1)^{i+j}\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2N} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,N} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,N} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{N,j-1} & a_{N,j+1} & \cdots & a_{NN} \end{pmatrix}.$$

2.28 Lemma. *Suppose that $\Omega \subseteq \mathbb{R}^N$ is open and $f \in C^2(\Omega, \mathbb{R}^N)$. We define $d_{ij}(x) := \det_{ij}(\mathrm{D}f(x))$. Then*

$$\sum_{j=1}^{N} \partial_j d_{ij}(x) = 0 \qquad \textit{for all } i = 1, 2, \dots, N \textit{ and } x \in \Omega.$$

Proof. In this proof the symbol $\widehat{\cdot}$ denotes omission of the object represented by the dot. Fix i and denote

$$g_k := \partial_k (f^1, f^2, \dots, \widehat{f^i}, \dots, f^N)^T.$$

It follows that

$$d_{ij}(x) = (-1)^{i+j}\det(g_1, g_2, \dots, \widehat{g_j}, \dots, g_N).$$

The determinant is linear in every column. It follows that

$$\partial_j d_{ij}(x) = (-1)^{i+j}\sum_{k\neq j}\det(g_1, g_2, \dots, \widehat{g_j}, \dots, \partial_j g_k, \dots, g_N). \tag{2.26}$$

We set

$$c_{kj} := \det(\partial_j g_k, g_1, g_2, \dots, \widehat{g_j}, \dots, \widehat{g_k}, \dots, g_N).$$

Since $f \in C^2(\Omega, \mathbb{R}^N)$, we have $\partial_k g_j = \partial_j g_k$. This implies that $c_{kj} = c_{jk}$. Interchanging two neighboring columns in a determinant just changes its sign. Therefore,

$$\det(g_1, g_2, \dots, \widehat{g_j}, \dots, \partial_j g_k, \dots, g_N) = \begin{cases} (-1)^{k-1}c_{kj}, & k < j, \\ (-1)^{k-2}c_{kj}, & k > j. \end{cases} \tag{2.27}$$

We define $\sigma_{kj} = 1$ for $k < j$, $\sigma_{jj} = 0$ and $\sigma_{kj} = -1$ for $k > j$. With this notation (2.26) and (2.27) imply that

$$(-1)^{i+j}\partial_j d_{ij}(x) = \sum_{k<j}(-1)^{k-1}c_{kj} + \sum_{k>j}(-1)^{k-2}c_{kj} = \sum_{k=1}^{N}(-1)^{k-1}\sigma_{kj}c_{kj}.$$

Adding terms up with respect to j we obtain

$$\begin{aligned}(-1)^i \sum_{j=1}^N \partial_j d_{ij}(x) &= \sum_{j,k=1}^N (-1)^{k-1+j} \sigma_{kj} c_{kj} \\ &= \sum_{j,k=1}^N (-1)^{j-1+k} \sigma_{jk} c_{jk} && \text{change of index} \\ &= -\sum_{j,k=1}^N (-1)^{k-1+j} \sigma_{kj} c_{kj} && \sigma_{jk} = -\sigma_{kj},\ c_{jk} = c_{kj}.\end{aligned}$$

This implies that the sum is 0. □

2.29 Proposition. *Suppose that $\Omega \subseteq \mathbb{R}^N$ is open and bounded, $f \in C^2(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$ and $y \in \mathbb{R}^N \backslash f(\partial\Omega)$. Set $\rho := \operatorname{dist}(y, f(\partial\Omega))$. If $y_1, y_2 \in B_\rho(y)$ are regular values of f, then*

$$\deg(f, \Omega, y_1) = \deg(f, \Omega, y_2),$$

*where the degree "*deg*" is given by Definition 2.24.*

Proof. Using Lemma 2.26, it is sufficient to prove that

$$\int_\Omega (\eta_\delta(f(x) - y_2) - \eta_\delta(f(x) - y_1)) J_f(x) \, \mathrm{d}x = 0 \tag{2.28}$$

for $\delta > 0$ small. The idea is to express $(\eta_\delta(f(x) - y_2) - \eta_\delta(f(x) - y_1)) J_f(x)$ as the divergence of a function v with compact support in Ω.

On exercise Sheet 2, no. 2 it was proved that for $\delta > 0$ the function

$$w(x) := (y_1 - y_2) \int_0^1 \eta_\delta(x - (1-t)y_1 - ty_2) \, \mathrm{d}t.$$

satisfies

$$\operatorname{div} w(x) = \eta_\delta(x - y_2) - \eta_\delta(x - y_1). \tag{2.29}$$

Unfortunately, this does not imply that $w \circ f$ has divergence $(\eta_\delta(f(x) - y_1) - \eta_\delta(f(x) - y_2)) J_f(x)$.

We define $\rho_1 := \max\{\, |y - y_1|, |y - y_2| \,\} < \rho$ and pick $0 < \delta_0 < \rho - \rho_1$. For $\delta \in (0, \delta_0]$, $t \in [0,1]$ and $x \in \mathbb{R}^N$ with $|x - y| > \rho_1 + \delta_0$ we obtain that

$$\begin{aligned}|x - (1-t)y_1 - ty_2| &\geq |x - y| - ((1-t)|y - y_1| + t|y - y_2|) \\ &> \rho_1 + \delta_0 - \rho_1 \\ &\geq \delta.\end{aligned}$$

Then $w(x) = 0$. This shows that

$$\operatorname{supp}(w) \subseteq \overline{B}_{\rho_1 + \delta_0}(y) \subseteq B_\rho(y) \subseteq \mathbb{R}^N \backslash f(\partial\Omega) \qquad \text{if } \delta \leq \delta_0. \tag{2.30}$$

Fixing $\delta \in (0, \delta_0]$ we define $d_{ij} := \det_{ij}(\mathrm{D}f)$ as in Lemma 2.28 and $v \in C^1(\Omega, \mathbb{R}^N)$ by

$$v^j(x) := \sum_{i=1}^{N} w^i(f(x)) d_{ij}(x).$$

By (2.30), $\operatorname{dist}(\operatorname{supp}(w \circ f), \partial\Omega) > 0$ and hence $\operatorname{supp}(v)$ is compact. Extending v to $\mathbb{R}^N$ by 0 gives $v \in C^1_{\mathrm{c}}(\mathbb{R}^N, \mathbb{R}^N)$. We calculate

$$\partial_j v^j(x) = \sum_{i=1}^{N} \sum_{k=1}^{N} \partial_k w^i(f(x)) \partial_j f^k(x) d_{ij}(x) + \sum_{i=1}^{N} w^i(f(x)) \partial_j d_{ij}(x). \tag{2.31}$$

It remains to sum up this expression with respect to j. For $i = k$ in the first sum we have

$$\sum_{j=1}^{N} d_{ij}(x) \partial_j f^k(x) = \sum_{j=1}^{N} d_{ij}(x) \partial_j f^i(x) = J_f(x)$$

(developing the determinant of $\mathrm{D}f(x)$ by the i-th row). For $i < k$ we observe that

$$\sum_{j=1}^{N} d_{ij}(x) \partial_j f^k(x) = \det \begin{pmatrix} \partial_1 f^1(x) & \dots & \partial_N f^1(x) \\ \vdots & & \vdots \\ \partial_1 f^{i-1}(x) & \dots & \partial_N f^{i-1}(x) \\ \partial_1 f^k(x) & \dots & \partial_N f^k(x) \\ \partial_1 f^{i+1}(x) & \dots & \partial_N f^{i+1}(x) \\ \vdots & & \vdots \\ \partial_1 f^{k-1}(x) & \dots & \partial_N f^{k-1}(x) \\ \partial_1 f^k(x) & \dots & \partial_N f^k(x) \\ \partial_1 f^{k+1}(x) & \dots & \partial_N f^{k+1}(x) \\ \vdots & & \vdots \\ \partial_1 f^N(x) & \dots & \partial_N f^N(x) \end{pmatrix} = 0.$$

Similarly one obtains that $\sum_{j=1}^{N} d_{ij}(x) \partial_j f^k(x) = 0$ if $i > k$. Summing up, this implies

$$\sum_{j=1}^{N} d_{ij}(x) \partial_j f^k(x) = \delta_{ik} J_f(x) \qquad \text{for all } i, k = 1, 2, \dots, N. \tag{2.32}$$

Joining (2.31) with Lemma 2.28 and (2.32) leads to

$$\begin{aligned} \operatorname{div} v(x) &= \sum_{i=1}^{N} \sum_{k=1}^{N} \partial_k w^i(f(x)) \sum_{j=1}^{N} \partial_j f^k(x) d_{ij}(x) + \sum_{i=1}^{N} w^i(f(x)) \sum_{j=1}^{N} \partial_j d_{ij}(x) \\ &= \sum_{i=1}^{N} \sum_{k=1}^{N} \partial_k w^i(f(x)) \delta_{ik} J_f(x) \\ &= J_f(x) \sum_{i=1}^{N} \partial_i w^i(f(x)) \\ &= J_f(x) \operatorname{div} w(f(x)). \end{aligned}$$

Recall that $\int_{\mathbb{R}^N} \operatorname{div} v = 0$ since $v \in C^1_{\mathrm{c}}(\mathbb{R}^N, \mathbb{R}^N)$. Using (2.29) this proves (2.28). $\square$

2.3.2 Singular Values and Functions in C^1

2.30 Lemma. *Suppose that $A, B \in \mathcal{L}(\mathbb{R}^N)$ and A is invertible. If moreover*

$$\|A - B\| < \frac{1}{\|A^{-1}\|} \tag{2.33}$$

holds true then $\operatorname{sgn}(\det A) = \operatorname{sgn}(\det B)$.

Proof. Define $\gamma \in C([0,1], \mathcal{L}(\mathbb{R}^N))$ by $\gamma(t) := (1-t)A + tB$. Observe that $\gamma(t) = (I - (A - \gamma(t))A^{-1})A$. By (2.33) we know that $\|(A - \gamma(t))A^{-1}\| \le t\|A - B\|\,\|A^{-1}\| < 1$. The Neumann series implies that $\gamma(t)^{-1}$ exists and that

$$\gamma(t)^{-1} = A^{-1}(I - (A - \gamma(t))A^{-1})^{-1} = A^{-1} \sum_{k=0}^{\infty} ((A - \gamma(t))A^{-1})^k.$$

Therefore, $\det(\gamma(t)) \neq 0$ for $t \in [0,1]$. The continuity of the determinant and the intermediate value theorem imply that $\operatorname{sgn}(\det A) = \operatorname{sgn}(\det \gamma(0)) = \operatorname{sgn}(\det \gamma(1)) = \operatorname{sgn}(\det B)$. $\square$

2.31 Lemma. *Suppose that $\Omega \subseteq \mathbb{R}^N$ is open and bounded, $f \in C^1(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$ and $y \in \mathbb{R}^N \backslash f(\partial\Omega \cup K_f)$. Then there are a compact set $K \subseteq \Omega$ and $r > 0$ such that for all $g \in C^1(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$ with*

$$\|f - g\|_{C(\overline{\Omega}, \mathbb{R}^N)} + \|\mathrm{D}f - \mathrm{D}g\|_{C(K, \mathcal{L}(\mathbb{R}^N))} \le r \tag{2.34}$$

it holds true that $y \in \mathbb{R}^N \backslash g(\partial\Omega \cup K_g)$ and

$$\deg(f, \Omega, y) = \deg(g, \Omega, y),$$

where the degree "$\deg$*" is given by Definition 2.24.*

Proof. Suppose that $f^{-1}(y) = \{ x_1, x_2, \dots, x_n \}$. We define the linear isomorphisms $A_k := \mathrm{D}f(x_k)$. There is $r_1 > 0$ such that the balls $\overline{B}_{r_1}(x_k) \subseteq \Omega$ are disjoint and that

$$|f(x) - y - A_k[x - x_k]| \le \frac{r_1}{2\|A_k^{-1}\|} \tag{2.35}$$

and

$$\|\mathrm{D}f(x) - A_k\| \le \frac{1}{4\|A_k^{-1}\|} \tag{2.36}$$

for all $k = 1, 2, \dots, n$ and $x \in \overline{B}_{r_1}(x_k)$. This follows because y is a regular value of f and by the continuous differenciability of f. We set $W := \bigcup_{k=1}^n B_{r_1}(x_k)$.

We prove that if $g \in C^1(\Omega, \mathbb{R}^N)$ satisfies

$$\|f - g\|_{C(\overline{W},\mathbb{R}^N)} \le \frac{r_1}{2\max_{k=1}^n \|A_k^{-1}\|} \tag{2.37}$$

and

$$\sup_{x\in\overline{W}} \|\mathrm{D}f(x) - \mathrm{D}g(x)\|_{\mathcal{L}(\mathbb{R}^N)} \le \frac{1}{4\max_{k=1}^n \|A_k^{-1}\|}, \tag{2.38}$$

then for all $k = 1, 2, \ldots, n$

$$\operatorname{sgn} J_g(x) = \operatorname{sgn} J_f(x_k) \qquad \text{for } x \in \overline{B}_{r_1}(x_k) \tag{2.39}$$

and

$$g \text{ has precisely one } y\text{-point in } \overline{B}_{r_1}(x_k). \tag{2.40}$$

To prove (2.39) we fix k, $x \in \overline{B}_{r_1}(x_k)$, $A := \mathrm{D}f(x)$ and $B := \mathrm{D}g(x)$, and calculate, using (2.36) and (2.38):

$$\|A_k - B\| \le \|A_k - A\| + \|A - B\| \le \frac{1}{2\|A_k^{-1}\|}.$$

Lemma 2.30 implies that $\operatorname{sgn} J_g(x) = \operatorname{sgn} J_f(x_k)$.

To prove (2.40) we fix k and define $\varphi\colon \overline{B}_{r_1}(x_k) \to \mathbb{R}^N$ by

$$\varphi(x) := x - A_k^{-1}(g(x) - y).$$

It follows that $\varphi(x) = x$ if and only if $g(x) = y$. If $x \in \overline{B}_{r_1}(x_k)$, then we have by (2.35) and (2.37)

$$|\varphi(x) - x_k| \le \|A_k^{-1}\|\big(|A_k[x - x_k] + y - f(x)| + |f(x) - g(x)|\big) \le r_1,$$

that is, $\varphi(\overline{B}_{r_1}(x_k)) \subseteq \overline{B}_{r_1}(x_k)$. Moreover,

$$\|\mathrm{D}\varphi(x)\| = \|I - A_k^{-1}\mathrm{D}g(x)\| \le \|A_k^{-1}\|\big(\|A_k - \mathrm{D}f(x)\| + \|\mathrm{D}f(x) - \mathrm{D}g(x)\|\big) \le \frac{1}{2}$$

by (2.36) and (2.38). For $x_1, x_2 \in \overline{B}_{r_1}(x_k)$ this implies that

$$|\varphi(x_1) - \varphi(x_2)| \le \int_0^1 |\mathrm{D}\varphi((1-t)x_1 + tx_2)[x_2 - x_1]|\,\mathrm{d}t \le \frac{1}{2}|x_1 - x_2|.$$

Then φ is a strict contraction in $\overline{B}_{r_1}(x_k)$ and Banach's fixed point theorem implies that there is precisely one fixed point of φ in $\overline{B}_{r_1}(x_k)$. This gives (2.40).

We set

$$r := \min\left\{ \frac{r_1}{2\max_{k=1}^n \|A_k^{-1}\|},\ \frac{1}{4\max_{k=1}^n \|A_k^{-1}\|},\ \frac{1}{2}\operatorname{dist}(y, f(\overline{\Omega}\backslash W)) \right\}.$$

Then $r > 0$. If $g \in C^1(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$ satisfies (2.34) for this r and with $K := \overline{W}$, then (2.37) and (2.38) are satisfied and imply (2.39) and (2.40). The definition of r implies that $g^{-1}(y) \subseteq W$. For $k = 1, 2, \ldots, n$ denote by z_k the unique y-point of g in $\overline{B}_{r_1}(x_k)$. Hence $g^{-1}(y) = \{ z_1, z_2, \ldots, z_n \}$, and y is a regular value of g, by (2.39) and because y is a regular value of f. Moreover, (2.39) implies that

$$\deg(f, \Omega, y) = \sum_{k=1}^{n} \operatorname{sgn} J_f(x_k) = \sum_{k=1}^{n} \operatorname{sgn} J_g(z_k) = \deg(g, \Omega, y). \qquad \square$$

The difference of the following proposition with Proposition 2.29 is that we only suppose that f is in C^1.

2.32 Proposition. *Suppose that $\Omega \subseteq \mathbb{R}^N$ is open and bounded, $f \in C^1(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$ and $y \in \mathbb{R}^N \backslash f(\partial\Omega)$. We set $\rho := \operatorname{dist}(y, f(\partial\Omega))$. If $y_1, y_2 \in B_\rho(y)$ are regular values of f, then*

$$\deg(f, \Omega, y_1) = \deg(f, \Omega, y_2),$$

*where the degree "*deg*" is given by Definition 2.24.*

Proof. Suppose that K_i and r_i are given by Lemma 2.31 with respect to y_i, $i = 1, 2$. Suppose that $\rho_1 \in (0, \rho)$ is such that $y_1, y_2 \in B_{\rho_1}(y)$. Smoothing the components of f and continuously extending them to $\mathbb{R}^N$ with Theorem 2.9, Proposition 2.16 provides $g \in C^\infty(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$ such that (2.34) holds true, where we define $K := K_1 \cup K_2$ and $r := \min\{r_1, r_2\}$. Moreover, we may assume that

$$\|f - g\|_{C(\overline{\Omega}, \mathbb{R}^N)} \leq \rho - \rho_1. \tag{2.41}$$

Then $\rho_1 \leq \operatorname{dist}(y, g(\partial\Omega))$. Using Lemma 2.31 and Proposition 2.29 we obtain

$$\deg(f, \Omega, y_1) = \deg(g, \Omega, y_1) = \deg(g, \Omega, y_2) = \deg(f, \Omega, y_2). \qquad \square$$

Using Proposition 2.32 and Sard's Lemma (Proposition 2.13) we may make the

2.33 Definition. Suppose that $\Omega \subseteq \mathbb{R}^N$ is open and bounded, $f \in C^1(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$ and $y \in \mathbb{R}^N \backslash f(\partial\Omega)$. We set $\rho := \operatorname{dist}(y, f(\partial\Omega))$ and define

$$\deg(f, \Omega, y) := \deg(f, \Omega, y_1),$$

where $y_1 \in B_\rho(y)$ is any regular value of f and $\deg(f, \Omega, y_1)$ is given by Definition 2.24.

2.34 Remark. If $f^{-1}(y) = \varnothing$, then $f^{-1}(y_1) = \varnothing$ for $y_1 \in \mathbb{R}^N \backslash f(K_f)$ that satisfies $|y_1 - y| < \operatorname{dist}(y, f(\overline{\Omega}))$. Definition 2.24 and Definition 2.33 imply that $\deg(f, \Omega, y) = \deg(f, \Omega, y_1) = 0$.

2.3.3 Approximation of Continuous Functions

For the extension of this definition to maps that are only continuous we need the

2.35 Proposition. *Suppose that $\Omega \subseteq \mathbb{R}^N$ is open and bounded, $f \in C(\overline{\Omega}, \mathbb{R}^N)$ and $y \in \mathbb{R}^N \setminus f(\partial\Omega)$. We set $\rho := \operatorname{dist}(y, f(\partial\Omega))$. If $g_1, g_2 \in C^1(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$ satisfy $\|f - g_i\|_{C(\overline{\Omega}, \mathbb{R}^N)} < \rho$ for $i = 1, 2$, then*

$$\deg(g_1, \Omega, y) = \deg(g_2, \Omega, y),$$

where "deg" is given by Definition 2.33.

Proof. Put $J := [0, 1]$ and define $h(t, x) := (1-t)g_1(x) + tg_2(x)$. We will use the notation $h_t := h(t, \cdot)$. For $t \in J$ and $x \in \partial\Omega$ we have

$$\begin{aligned}|h(t,x) - y| \geq |f(x) - y| - \big((1-t)|g_1(x) - f(x)| + t|g_2(x) - f(x)|\big) \\ > \rho - ((1-t)\rho + t\rho) = 0.\end{aligned}$$

Then by Definition 2.33, $\varphi(t) := \deg(h_t, \Omega, y)$ is well defined for all $t \in J$. We need to show that φ is constant. Since φ takes values in $\mathbb{Z}$ it is sufficient to show that

(2.42) $\qquad \varphi$ is locally constant on J

since this implies continuity and hence constancy of φ.

Fix $t_0 \in J$.

Case 1: $h_{t_0}^{-1}(y) = \varnothing$. Set $\delta := \operatorname{dist}(y, h_{t_0}(\overline{\Omega}))$. If $|t - t_0| \leq \delta/(2\rho)$, then

$$\begin{aligned}|h(t,x) - y| &\geq |h(t_0, x) - y| - |h(t_0, x) - h(t, x)| \\ &\geq \delta - |t - t_0| \, \|g_1 - g_2\|_\infty \\ &\geq \delta - \frac{\delta}{2\rho}(\|g_1 - f\|_\infty + \|g_2 - f\|_\infty) \\ &> 0\end{aligned}$$

for all $x \in \overline{\Omega}$. Therefore, $h_t^{-1}(y) = \varnothing$, and by Remark 2.34 $\varphi(t) = \deg(h_t, \Omega, y) = 0$ for $t \in [t_0 - \delta/(2\rho), t_0 + \delta/(2\rho)] \cap J$.

Case 2: $y \in h_{t_0}(\Omega)$ is a regular value of h_{t_0}. Write $h_{t_0}^{-1}(y) = \{x_1, x_2, \ldots, x_n\}$. The implicit function theorem implies that there are $r, s_1 > 0$ and functions $z_k \in C^1((t_0 - s_1, t_0 + s_1), B_r(x_k))$ such that $h_t^{-1}(y) \cap B_r(x_k) = \{z_k(t)\}$ and $z_k(t_0) = x_k$, for $t \in (t_0 - s_1, t_0 + s_1)$ and $k = 1, 2, \ldots, n$. Without restriction suppose that the balls $B_r(x_k)$ are disjoint and that $\operatorname{sgn} J_{h_{t_0}}(x) = \operatorname{sgn} J_{h_{t_0}}(x_k)$ for $x \in \overline{B}_r(x_k)$, using Lemma 2.30.

Define $V := \bigcup_{k=1}^n B_r(x_k)$ and $\delta := \operatorname{dist}(y, h_{t_0}(\overline{\Omega} \setminus V)) > 0$. As in Case 1 one proves that

$$h_t^{-1}(y) \cap (\overline{\Omega} \setminus V) = \varnothing \qquad \text{for } |t - t_0| \leq \frac{\delta}{2\rho}.$$

The function $(t, x) \mapsto J_{h_t}(x)$ is uniformly continuous in $J \times \overline{V}$, and $J_{h_{t_0}}$ has no zero in $\overline{V}$. There is $s_2 > 0$ such that $J_{h_t}(x) \neq 0$ for $(t, x) \in (t_0 - s_2, t_0 + s_2) \times \overline{V}$.

With $\mu := \min\{s_1, s_2, \delta/(2\rho)\}$ we obtain that

$$h_t^{-1}(y) = \{\, z_1(t), z_2(t), \dots, z_n(t) \,\}$$

and that $\operatorname{sgn} J_{h_t}(z_k(t)) = \operatorname{sgn} J_{h_{t_0}}(x_k) \neq 0$ for all $t \in (t_0 - \mu, t_0 + \mu) \cap J$ and all k. It follows for $t \in (t_0 - \mu, t_0 + \mu) \cap J$ that

$$\varphi(t) = \sum_{k=1}^{n} \operatorname{sgn} J_{h_t}(z_k(t)) = \sum_{k=1}^{n} \operatorname{sgn} J_{h_{t_0}}(x_k) = \varphi(t_0).$$

Case 3: $y \in h_{t_0}(\Omega)$ is a singular value of h_{t_0}. Set $\delta := \operatorname{dist}(y, h_{t_0}(\partial\Omega))$ and pick a regular value $y_1 \in B_\delta(y)$ of h_{t_0}. As in Case 2 it follows that there is $s > 0$ such that y_1 is a regular value of h_t and that $\deg(h_t, \Omega, y_1) = \deg(h_{t_0}, \Omega, y_1)$ for $t \in (t_0 - s, t_0 + s)$. Taking s sufficiently small we may assume that $|y - y_1| < \operatorname{dist}(y, h_t(\partial\Omega))$ for these values of t. Using Definition 2.33 we obtain that $\varphi(t) = \varphi(t_0)$ if $t \in (t_0 - s, t_0 + s)$. This concludes the proof of (2.42). □

By Proposition 2.35 and Proposition 2.16 we may make the following

2.36 Definition. Suppose that $\Omega \subseteq \mathbb{R}^N$ is open and bounded, $f \in C(\overline{\Omega}, \mathbb{R}^N)$ and $y \in \mathbb{R}^N \setminus f(\partial\Omega)$. Define

$$\deg(f, \Omega, y) := \deg(g, \Omega, y),$$

where $g \in C^1(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$ satisfies $\|f - g\|_{C(\overline{\Omega}, \mathbb{R}^N)} < \operatorname{dist}(y, f(\partial\Omega))$ and $\deg(g, \Omega, y)$ is given by Definition 2.33.

2.37 Remark. Suppose that Ω, f and y are as in the preceding definition. We may calculate the degree directly with Definition 2.24 (using a regular value) as follows: Set $\rho := \operatorname{dist}(y, f(\partial\Omega))$ and pick $g \in C^1(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$ and a regular value $z \in \mathbb{R}^N \setminus g(\partial\Omega)$ of g such that $\|f - g\|_\infty \leq \rho/3$ and $|y - z| \leq \rho/3$. It follows that $\|f - g\|_\infty < \operatorname{dist}(y, f(\partial\Omega))$ and

$$|y - z| \leq \rho/3 < 2\rho/3 \leq \operatorname{dist}(y, f(\partial\Omega)) - \|f - g\|_\infty \leq \operatorname{dist}(y, g(\partial\Omega)).$$

By Definition 2.33 and Definition 2.24 we obtain

$$\deg(f, \Omega, y) = \deg(g, \Omega, y) = \deg(g, \Omega, z). \qquad (2.43)$$

2.38 Theorem. *The degree given in Definition 2.36 satisfies* (D1)–(D3).

Proof. **(D1)** Since "id" is differentiable and 0 a regular value, Definition 2.24 implies the claim.

(D2) Define $A := \overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)$ and $\rho := \operatorname{dist}(y, f(A)) > 0$. By extension and smoothing (Theorem 2.9 and Proposition 2.16) we find $g \in C^\infty(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$ and a regular value $z \in \mathbb{R}^N \backslash g(\partial\Omega)$ of g such that $\|f - g\|_\infty \leq \rho/3$ and $|y - z| \leq \rho/3$. Observe that $\partial\Omega \subseteq A$ implies $\rho \leq \operatorname{dist}(y, f(\partial\Omega))$. Therefore, Remark 2.37 implies

$$\deg(f, \Omega, y) = \deg(g, \Omega, z). \tag{2.44}$$

Since $g^{-1}(z) \cap A = \varnothing$, Definition 2.24 gives

$$\begin{aligned}\deg(g, \Omega, z) &= \sum_{x \in g^{-1}(z)} \operatorname{sgn} J_g(x) \\ &= \sum_{x \in g^{-1}(z) \cap \Omega_1} \operatorname{sgn} J_g(x) + \sum_{x \in g^{-1}(z) \cap \Omega_2} \operatorname{sgn} J_g(x) \\ &= \deg(g, \Omega_1, z) + \deg(g, \Omega_2, z).\end{aligned} \tag{2.45}$$

The inclusions $\partial\Omega_i \subseteq A$ imply that $\rho \leq \operatorname{dist}(y, f(\partial\Omega_i))$ for $i = 1, 2$. Using Remark 2.37 once more we obtain

$$\deg(f, \Omega_i, y) = \deg(g, \Omega_i, z) \qquad \text{for } i = 1, 2. \tag{2.46}$$

Joining Equations (2.44), (2.45) and (2.46) we conclude.

(D3) Suppose that $J := [0, 1]$, $h \in C(J \times \overline{\Omega}, \mathbb{R}^N)$ is a homotopy and $y \in C(J, \mathbb{R}^N)$ is such that $y(t) \notin h(t, \partial\Omega)$ for all $t \in J$. Define $\rho(t) := \operatorname{dist}(y(t), h(t, \partial\Omega)) > 0$ for $t \in J$. We prove that $\rho_0 := \min \rho(J) > 0$. By contradiction, suppose that there is $(t_n) \subseteq J$ such that $\rho(t_n) \to 0$ as $n \to \infty$. For every n there is $x_n \in \partial\Omega$ with $|y(t_n) - h(t_n, x_n)| \leq \rho(t_n) + 1/n$, by the definition of ρ. Since $\partial\Omega$ and J are compact, passing to a subsequence we may assume that $t_n \to t^* \in J$ and $x_n \to x^* \in \partial\Omega$ as $n \to \infty$. The continuity of y and h imply that

$$0 < \rho(t^*) \leq |y(t^*) - h(t^*, x^*)| = \lim_{n\to\infty} |y(t_n) - h(t_n, x_n)| \leq \lim_{n\to\infty} (\rho(t_n) + 1/n) = 0,$$

a contradiction.

By extension and smoothing we find $H \in C^\infty(\mathbb{R}^{N+1}, \mathbb{R}^N)$ such that $\|h - H\|_{C(J \times \overline{\Omega}, \mathbb{R}^N)} \leq \rho_0/4$. As usual, we write $h_t := h(t, \cdot)$ and $H_t := H(t, \cdot)$. It follows that

$$\|h_t - H_t\|_{C(\overline{\Omega}, \mathbb{R}^N)} \leq \frac{\rho_0}{4} < \rho_0 \leq \rho(t) = \operatorname{dist}(y(t), h_t(\partial\Omega))$$

and, by Definition 2.36,

$$\deg(h_t, \Omega, y(t)) = \deg(H_t, \Omega, y(t)) \qquad \text{for all } t \in J. \tag{2.47}$$

Observe that

$$\operatorname{dist}(y(t), H_t(\partial\Omega)) \geq \frac{3\rho_0}{4} \qquad \text{for all } t \in J. \tag{2.48}$$

Since H is uniformly continuous in $J \times \overline{\Omega}$ and y is uniformly continuous in J there is $\delta > 0$ such that $\|H_s - H_t\|_{C(\overline{\Omega}, \mathbb{R}^N)} \leq \rho_0/4$ and $|y(s) - y(t)| \leq \rho_0/4$ for $s, t \in J$ with

$|s-t| \le \delta$. Fix such s, t and observe that $V := B_{\rho_0/4}(y(s)) \cap B_{\rho_0/4}(y(t))$ is open and not empty. Sard's Lemma implies that $A := H_s(K_{H_s}) \cup H_t(K_{H_t})$ has zero measure, that is, $\mathbb{R}^N \backslash A$ is dense in $\mathbb{R}^N$. In consequence, there is $z \in V \backslash A$, i.e. $z \in V$ is a regular value of H_s and of H_t.

Observe that (2.48) yields

$$|z - y(s)| \le \frac{\rho_0}{4} < \frac{3\rho_0}{4} \le \operatorname{dist}(y(s), H_s(\partial\Omega)).$$

Now Definition 2.33 yields

$$\deg(H_s, \Omega, y(s)) = \deg(H_s, \Omega, z). \tag{2.49}$$

Similarly we obtain

$$\deg(H_t, \Omega, y(t)) = \deg(H_t, \Omega, z). \tag{2.50}$$

From (2.48) and $|z - y(s)| \le \rho_0/4$ we infer $\operatorname{dist}(z, H_s(\partial\Omega)) \ge \frac{\rho_0}{2}$. Hence the inequality $\|H_s - H_t\|_{C(\overline{\Omega}, \mathbb{R}^N)} \le \frac{\rho_0}{4}$ (definition of δ, s and t) and Proposition 2.35 imply that

$$\deg(H_s, \Omega, z) = \deg(H_t, \Omega, z). \tag{2.51}$$

By the preceding equations we deduce

$$\begin{aligned} \deg(h_s, \Omega, y(s)) &= \deg(H_s, \Omega, y(s)), && \text{by (2.47)}, \\ &= \deg(H_s, \Omega, z), && \text{by (2.49)}, \\ &= \deg(H_t, \Omega, z), && \text{by (2.51)}, \\ &= \deg(H_t, \Omega, y(t)), && \text{by (2.50)}, \\ &= \deg(h_t, \Omega, y(t)), && \text{by (2.47)}. \end{aligned} \tag{2.52}$$

Define $\varphi \colon J \to \mathbb{Z}$ by $\varphi(t) := \deg(h_t, \Omega, y(t))$. Equation (2.52) implies that φ is locally constant, that is, $\varphi \in C(J)$. Therefore, φ is constant on J. □

3 Properties and Applications of the Degree

3.1 Basic Properties and Applications

For $N \in \mathbb{N}$ we define

(3.1) $\mathcal{A}_N := \{ (f, \Omega, y) \mid \Omega \subseteq \mathbb{R}^N$ open and bounded, $f \in C(\overline{\Omega}, \mathbb{R}^N),\ y \in \mathbb{R}^N \backslash f(\partial\Omega) \}$,

the set of admissible triplets in dimension N. Definition 2.36 defines the degree as a map $\deg\colon \mathcal{A}_N \to \mathbb{Z}$ with the properties (D1)–(D3).

3.1 Proposition. *The degree satisfies the following additional properties:*

(D4) $\deg(f, \Omega, y) \neq 0$ *implies that* $f^{-1}(y) \neq \varnothing$.

(D5) $\deg(\cdot, \Omega, y)$ *and* $\deg(f, \Omega, \cdot)$ *are constant functions in* $B_\rho(f; C(\overline{\Omega}, \mathbb{R}^N))$ *and* $B_\rho(y; \mathbb{R}^N)$, *respectively, where* $\rho := \operatorname{dist}(y, f(\partial\Omega)) > 0$. *Moreover,* $\deg(f, \Omega, \cdot)$ *is constant in every connected component of* $\mathbb{R}^N \backslash f(\partial\Omega)$.

(D6) $\deg(g, \Omega, y) = \deg(f, \Omega, y)$ *if* $g|_{\partial\Omega} = f|_{\partial\Omega}$.

(D7) $\deg(f, \Omega, y) = \deg(f, \Omega_1, y)$ *for all open subsets* Ω_1 *of* Ω *such that* $y \notin f(\overline{\Omega} \backslash \Omega_1)$.

Proof. **(D4)** We showed at the beginning of the proof of Proposition 2.18 that $f^{-1}(y) = \varnothing$ implies that $\deg(f, \Omega, y) = 0$.

(D5) The first claim was proved on Exercise Sheet 2, N°1. For the second claim, let y_1 and y_2 belong to the same connected component. There is a path $y(t)$ from y_1 to y_2 in $\mathbb{R}^N \backslash f(\partial\Omega)$ since this set is locally path connected. Property (D3) implies that $\deg(f, \Omega, y_1) = \deg(f, \Omega, y_2)$.

(D6) This is a special case of Exercise Sheet 2, N°1(ii).

(D7) This is a consequence of properties (D4) and (D2), where $\Omega_2 = \varnothing$. □

3.2 Theorem (Brouwer)**.** *Suppose that* $K \subseteq \mathbb{R}^N$ *is compact, convex and not empty, and suppose that* $f\colon K \to K$ *is continuous. Then* f *has a fixed point. The same holds true if* K *is homeomorphic to a subset of* $\mathbb{R}^N$ *that is compact, convex and not empty.*

Proof. We prove that $x - f(x)$ has a zero in K. First suppose that $K = \overline{B}_r$ for $r > 0$. If there is $x \in S_r$ such that $f(x) = x$ then we conclude. If not then $x - f(x) \neq 0$ for all

$x \in S_r$. The homotopy $h(t,x) = (1-t)x + t(x - f(x)) = x - tf(x)$ of $\mathrm{id}_{\overline{B}_r}$ to $\mathrm{id}_{\overline{B}_r} - f$ satisfies for $x \in S_r$ and $t \in [0,1)$:

$$|h(t,x)| \geq |x| - t|f(x)| \geq (1-t)r > 0.$$

For $t = 1$ and $x \in S_r$ we have that $h(t,x) = x - f(x) \neq 0$. Then $0 \in B_r$, (D3) and (D1) yield $\deg(\mathrm{id} - f, B_r, 0) = 1$. Hence (D4) implies that $x - f(x) = 0$ for some $x \in B_r$.

More generally, if $K \subseteq \mathbb{R}^N$ is compact, convex and not empty we extend f to $\mathbb{R}^N$ continuously, as in Theorem 2.9, and denote the extension by g. The convexity of K implies that $g(\mathbb{R}^N) \subseteq K$. We find $r > 0$ sufficient large such that $K \subseteq \overline{B}_r$. Hence $g(\overline{B}_r) \subseteq K \subseteq \overline{B}_r$. By the first part there is a fixed point x of g in $\overline{B}_r$, that is in K. Since $g|_K = f$, x is a fixed point of f.

Suppose that K a metric space, $f\colon K \to K$ continuous, $A \subseteq \mathbb{R}^N$ compact, convex and not empty, and suppose that $\varphi\colon K \to A$ is a homeomorphism. Then $g := \varphi \circ f \circ \varphi^{-1}$ has a fixed point x in A by the preceding paragraph, and hence $\varphi^{-1}(x)$ is a fixed point of f:

$$f(\varphi^{-1}(x)) = (\varphi^{-1} \circ \varphi \circ f \circ \varphi^{-1})(x) = (\varphi^{-1} \circ g)(x) = \varphi^{-1}(x). \qquad \square$$

3.3 Definition. Suppose that X is a metric space and $A \subseteq X$. A *retraction of X to A* is a continuous map $\varphi\colon X \to A$ such that $\varphi|_A = \mathrm{id}_A$. In this case A is a *retract of X.*

3.4 Example. $\varphi\colon \mathbb{R}^N \to \overline{B}_1$, given by

$$\varphi(x) := \begin{cases} x, & |x| \leq 1, \\ \frac{x}{|x|}, & |x| \geq 1, \end{cases}$$

is a retraction of $\mathbb{R}^N$ to $\overline{B}_1$.

3.5 Corollary (to the theorem of Brouwer)**.** *There is no retraction of $\overline{B}_r$ to S_r.*

Proof. Arguing by contradiction, suppose that φ is a retraction of $\overline{B}_r$ to S_r. Then $-\varphi$ has a fixed point x in $\overline{B}_r$ by Theorem 3.2. Since $\varphi(\overline{B}_r) \subseteq S_r$ it follows that $x \in S_r$, that is, $x \neq 0$ is also a fixed point of φ. It follows that $x = -\varphi(x) = -x$. Contradiction! □

3.6 Proposition (Hedgehog Theorem)**.** *Suppose that N is odd, $\Omega \subseteq \mathbb{R}^N$ open and bounded such that $0 \in \Omega$, and $f \in C(\partial\Omega, \mathbb{R}^N)$. Then there are $x \in \partial\Omega$ and $\lambda \in \mathbb{R}$ such that $f(x) = \lambda x$.*

Proof. We extend f continuously to $\overline{\Omega}$ using Theorem 2.9. Since N is odd we have $\deg(-\mathrm{id}, \Omega, 0) = -1$. If $\deg(f, \Omega, 0) \neq -1$ we define the linear homotopy $h(t,x) = (1-t)f(x) - tx$ from f to $-\mathrm{id}$. By (D3) there are $t_0 \in [0,1)$ and $x_0 \in \partial\Omega$ such that $h(t_0, x_0) = 0$. We obtain

$$f(x_0) = \frac{t_0}{1-t_0} x_0.$$

If $\deg(f, \Omega, 0) = -1$, use the linear homotopy from f to id, $\deg(\mathrm{id}, \Omega, 0) = 1$ and a similar argument. □

3.7 Remark. The name given to the previous theorem is to be undestood as follows: One cannot comb a hedgehog without a bald spot. To explain: For any $r > 0$ the ball $\overline{B}_r\mathbb{R}^3$ symbolizes the hedgehog. The spines on $S_r\mathbb{R}^3$ are given by a continuous vector field $f\colon S_r \to \mathbb{R}^3$. A bald spot is a zero of f. To comb means to put the spines into a position tangential to S_r, that is, such that $f(x) \cdot x = 0$ for all $x \in S_r$. If f is continuous and tangential to S_r, by Proposition 3.6 there are $x_0 \in S_r$ and $\lambda \in \mathbb{R}$ such that $f(x_0) = \lambda x_0$. Since $f(x_0)$ is tangential to S_r, $\lambda = 0$ and hence $f(x_0) = 0$.

3.8 Example. The hedgehog theorem is not valid in even dimension. For example, for $N = 2$ the vector field f on S_1 given by

$$f(x) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

is continuous, tangential and has no zero.

3.9 Proposition. *Suppose that* $f \in C(\mathbb{R}^N, \mathbb{R}^N)$ *satisfies*

$$\frac{f(x) \cdot x}{|x|} \to \infty \qquad \text{as } |x| \to \infty. \tag{3.2}$$

Then $f(\mathbb{R}^N) = \mathbb{R}^N$.

Proof. Fix $y \in \mathbb{R}^N$ and define $h(t, x) := (1 - t)f(x) + tx$. Suppose that $r \geq 2|y| + 1$ is large enough such that

$$\frac{f(x) \cdot x}{|x|} \geq 2|y| + 1 \qquad \text{for all } x \in S_r.$$

We calculate for $x \in S_r$:

$$\begin{aligned}(h(t, x) - y) \cdot x &= (1 - t)f(x) \cdot x + tr^2 - y \cdot x \\ &\geq (1 - t)(2|y| + 1)r + t(2|y| + 1)r - |y|r = (|y| + 1)r > 0.\end{aligned}$$

By (D1) and (D3) this implies that $\deg(f, B_r, y) = \deg(\mathrm{id}, B_r, y) = 1$, that is, $f^{-1}(y) \cap B_r \neq \varnothing$. □

3.2 Varying the Space

Suppose that E is a Banach space of finite dimension. Denote by $\mathcal{A}_E$ the *set of admissible triplets* (f, Ω, y) where $\Omega \subseteq E$ is open and bounded, $f \in C(\overline{\Omega}, E)$ and $y \in E \backslash f(\partial\Omega)$.

If E has the base $\Lambda := \{ x_1, x_2, \ldots, x_N \}$, we define the linear map $\varphi\colon E \to \mathbb{R}^N$ by

$$\varphi(x_k) := e_k.$$

By definition, the elements e_k form the canonical base of $\mathbb{R}^N$. It follows that $\varphi \in \mathcal{L}(E, \mathbb{R}^N)$ is a linear isomorphism and a homeomorphism of E and $\mathbb{R}^N$. We call φ the *canonical isomorphism between* E *and* $\mathbb{R}^N$ *with respect to the base* Λ.

3.10 Proposition. *Suppose that E is a Banach space with $\dim E = N < \infty$ and let φ denote the canonical isomorphism between E and $\mathbb{R}^N$ with respect to a base of E. Suppose that $(f, \Omega, y) \in \mathcal{A}_E$. Then $(\varphi \circ f \circ \varphi^{-1}, \varphi(\Omega), \varphi(y)) \in \mathcal{A}_N$ and we define*

$$\deg_E(f, \Omega, y) := \deg(\varphi \circ f \circ \varphi^{-1}, \varphi(\Omega), \varphi(y)).$$

Then $\deg_E$ does not depend on the selection of the base of E and satisfies the properties (D1)–(D7)*, replacing $\mathbb{R}^N$ by E.*

Proof. Is clear that $(\varphi \circ f \circ \varphi^{-1}, \varphi(\Omega), \varphi(y)) \in \mathcal{A}_N$ because φ is a homeomorphism and $\partial(\varphi(\Omega)) = \varphi(\partial\Omega)$. Suppose that $\Lambda_1 := \{ x_1, x_2, \ldots, x_N \}$ and $\Lambda_2 := \{ y_1, y_2, \ldots, y_N \}$ are two bases of E, and let φ_1 and φ_2 denote the canonical isomorphisms of E and $\mathbb{R}^N$ with respect to the bases Λ and Γ. We define $A := \varphi_2 \circ \varphi_1^{-1} \in \mathcal{L}(\mathbb{R}^N)$. Suppose that $\Omega_1 := \varphi_1(\Omega)$ and $\Omega_2 := \varphi_2(\Omega) = A(\Omega_1)$. Fix $g_1 \in C^1(\Omega_1, \mathbb{R}^N) \cap C(\overline{\Omega_1}, \mathbb{R}^N)$ and a regular value z_1 of g_1 such that with $g_2 := A \circ g_1 \circ A^{-1}$ and $z_2 := Az_1$ we have

$$\|\varphi_i \circ f \circ \varphi_i^{-1} - g_i\|_{C(\overline{\Omega_i}, \mathbb{R}^N)} \leq \frac{\operatorname{dist}(\varphi_i(y), \partial\Omega_i)}{3}, \tag{3.3}$$

$$|\varphi_i(y) - z_i| \leq \frac{\operatorname{dist}(\varphi_i(y), \partial\Omega_i)}{3}, \tag{3.4}$$

for $i = 1, 2$. Remark 2.37 implies that

$$\deg(\varphi_i \circ f \circ \varphi_i^{-1}, \varphi_i(\Omega), \varphi_i(y)) = \deg(g_i, \Omega_i, z_i). \tag{3.5}$$

Moreover, z_2 is a regular value of g_2.

Since $g_2^{-1}(z_2) = A(g_1^{-1}(z_1))$,

$$\operatorname{sgn} J_{g_2}(Ax) = \operatorname{sgn} \det(\mathrm{D}g_2(Ax)) = \operatorname{sgn} \det(A\mathrm{D}g_1(x)A^{-1}) = \operatorname{sgn} J_{g_1}(x)$$

if $x \in g_1^{-1}(z_1)$. This yields

$$\begin{aligned} \deg(g_2, \Omega_2, z_2) &= \sum_{x \in g_2^{-1}(z_2)} \operatorname{sgn} J_{g_2}(x) \\ &= \sum_{x \in A(g_1^{-1}(z_1))} \operatorname{sgn} J_{g_2}(x) \\ &= \sum_{x \in g_1^{-1}(z_1)} \operatorname{sgn} J_{g_2}(Ax) \\ &= \sum_{x \in g_1^{-1}(z_1)} \operatorname{sgn} J_{g_1}(x) \\ &= \deg(g_1, \Omega_1, z_1). \end{aligned} \tag{3.6}$$

Combining equations (3.5) and (3.6) we obtain that

$$\deg(\varphi_1 \circ f \circ \varphi_1^{-1}, \varphi_1(\Omega), \varphi_1(y)) = \deg(\varphi_2 \circ f \circ \varphi_2^{-1}, \varphi_2(\Omega), \varphi_2(y)).$$

Proving that this degree satisfies properties (D1)–(D7) is only a cuestion of applying the definition and the same properties for the usual degree. □

3.11 Proposition. *Let E denote a Banach space of finite dimension and F a subspace of E. Suppose moreover that $\Omega \subseteq E$ is open and bounded, $g \in C(\overline{\Omega}, F)$, $f \in C(\overline{\Omega}, E)$ is given by $f := \mathrm{id}_E - g$ and $y \in F \backslash f(\partial\Omega)$. Denoting $\Omega' := \Omega \cap F$ it holds true that $f(\overline{\Omega'}) \subseteq F$, $(f|_{\overline{\Omega'}}, \Omega', y) \in \mathcal{A}_F$ and*

$$\deg_E(f, \Omega, y) = \deg_F(f|_{\overline{\Omega'}}, \Omega', y). \tag{3.7}$$

Proof. Denote $h := f|_{\overline{\Omega'}} \in C(\overline{\Omega'}, F)$. Since

$$\partial_F \Omega' \subseteq \partial_E \Omega \cap F, \tag{3.8}$$

it follows that $(f|_{\overline{\Omega'}}, \Omega', y) \in \mathcal{A}_F$. By Proposition 3.10 we may assume that $E = \mathbb{R}^N$ and $F = \mathbb{R}^M \times \{0\}$ with $M \leq N$.

First suppose that $g \in C^1(\Omega, \mathbb{R}^M) \cap C(\overline{\Omega}, \mathbb{R}^M)$ and $y \in \mathbb{R}^M \backslash f(\partial\Omega)$ is a regular value of f. By the definition of f it follows that $f^{-1}(y) \subseteq \mathbb{R}^M$ and hence that

$$f^{-1}(y) = h^{-1}(y). \tag{3.9}$$

For $x \in f^{-1}(y)$ we have a representation of $\mathrm{D}f(x)$ by a matrix

$$A = \begin{pmatrix} I_M - B & C \\ 0 & I_{N-M} \end{pmatrix} \tag{3.10}$$

where

$$B := (\partial_j g^i(x))_{i,j \in \{1,2,\ldots,M\}} \in \mathbb{R}^{M \times M}$$

and

$$C := (\partial_j g^i(x))_{i \in \{1,2,\ldots,M\}, j \in \{M+1, M+2, \ldots, N\}} \in \mathbb{R}^{M \times (N-M)}.$$

Clearly

$$J_f(x) = \det(A) = \det(I_M - B) = J_h(x). \tag{3.11}$$

This implies that

$$\deg_E(f, \Omega, y) = \sum_{x \in f^{-1}(y)} \mathrm{sgn}\, J_f(x) = \sum_{x \in h^{-1}(y)} \mathrm{sgn}\, J_h(x) = \deg_F(h, \Omega', y).$$

In the irregular case we set $\rho := \mathrm{dist}(y, f(\partial\Omega))$ and approximate g by a map $g_1 \in C^1(\Omega, \mathbb{R}^M) \cap C(\overline{\Omega}, \mathbb{R}^M)$ such that $\|g - g_1\|_\infty \leq \rho/3$. Define $f_1 := \mathrm{id} - g_1$ and $h_1 := f_1|_{\overline{\Omega'}}$. It follows that $\|f - f_1\|_\infty \leq \rho/3$. By (3.8) we obtain

$$\|h - h_1\|_\infty \leq \frac{\rho}{3} \leq \mathrm{dist}(y, h(\partial\Omega'))/3.$$

Fix a regular value $y_1 \in \mathbb{R}^M$ of h_1 such that $|y - y_1| \leq \rho/3$, using the theorem of Sard. The formula (3.11), applied to f_1 and h_1 in place of f and h, implies that y_1 is a regular value of f_1. Moreover,

$$|y - y_1| \leq \frac{\rho}{3} \leq \mathrm{dist}(y, h(\partial\Omega'))/3.$$

Remark 2.37 and the preceding equations yield

$$\deg_E(f, \Omega, y) = \deg_E(f_1, \Omega, y_1) = \deg_F(h_1, \Omega', y_1) = \deg_F(h, \Omega', y)$$

and we conclude. □

4 The Degree in Infinite Dimensions

4.1 Introduction

In this section we will extend the topological degree to maps in Banach spaces. The motivation is that, to prove existence of solutions of differential equations we have to allow infinite dimensional spaces.

4.1 Example. Suppose that $f \in C(\mathbb{R}, \mathbb{R})$ is bounded. Consider the ordinary differential equation

$$\dot{u}(t) = f(u(t)), \qquad u(0) = u_0 \tag{4.1}$$

in an interval $J := [0, a]$, for $u_0 \in \mathbb{R}$. Write $\dot{u} := \mathrm{d}u/\mathrm{d}t$. If f is Lipschitz continuous then Picard-Lindelöf's Existence and Uniqueness Theorem applies.

Without Lipschitz continuity of f it still holds true for $u \in E := C(J, \mathbb{R})$ that u is a solution of (4.1) if and only if u is a solution of

$$u(t) = u_0 + \int_0^t f(u(s))\,\mathrm{d}s. \tag{4.2}$$

We define $K\colon E \to E$ by

$$K(u)(t) := u_0 + \int_0^t f(u(s))\,\mathrm{d}s.$$

Then $u \in E$ is a solution of (4.2) if and only if u is a fixed point of K, or equivalently, a zero of $\mathrm{id} - K$.

Suppose that $M := \sup f(\mathbb{R})$. If $K(u) = u$, then

$$|u(t)| \leq |u_0| + Ma \qquad \text{for } t \in J,$$

that is, $\|u\|_E \leq r := |u_0| + Ma$. Hence all fixed points of K are in $\overline{B}_r E$. To prove that there is a solution of (4.1) we need to define a topological degree for the map $\mathrm{id} - K$ in E, and prove that $\deg(\mathrm{id} - K, B_{r+\varepsilon}, 0) \neq 0$ for $\varepsilon > 0$.

The principal idea is to define a degree for maps of the type $\mathrm{id} - K$ in Banach spaces, where K has a "thin" image A in E. Then one restricts $\mathrm{id} - K$ to a subspace of finite dimension that approximates A, and uses the idea of Proposition 3.11.

4.2 Compactness in Banach Spaces

4.2 Definition. Suppose that X is a metric space. A subset $A \subseteq X$ is *relatively compact* if $\overline{A}$ is compact. A is *precompact* if for all $\varepsilon > 0$ there are points $x_1, x_2, \dots, x_n \in X$ such that

$$A \subseteq \bigcup_{k=1}^{n} B_\varepsilon(x_k).$$

4.3 Remark. Suppose that X, Y are metric spaces, $A \subseteq X$ and $f\colon X \to Y$ is continuous.

(a) A is relatively compact if and only if every sequence in A has a subsequence that converges in X.

(b) If A is relatively compact, then $f(A)$ is relatively compact.

(c) If X is complete, then A is relatively compact if and only if A is precompact.

4.4 Lemma. *Suppose that E is a normed space, $A \subseteq E$, $X \subseteq E$ is finite and $\varepsilon > 0$ is such that $A \subseteq X + B_\varepsilon$. If $\varepsilon_1 > \varepsilon$ then there is a finite set $Y \subseteq E$ such that* $\operatorname{conv}(A) \subseteq Y + B_{\varepsilon_1}$.

Proof. Suppose that $x \in \operatorname{conv}(A)$. Then there are $n \in \mathbb{N}$, $\lambda_k \in [0,1]$ and $x_k \in A$ for $k = 1, 2, \dots, n$ such that $\sum_{k=1}^{n} \lambda_k = 1$ and $x = \sum_{k=1}^{n} \lambda_k x_k$. By the hipothesis we can fix $y_k \in X$ such that $|x_k - y_k| < \varepsilon$ for $k = 1, 2, \dots, n$. It follows that

$$x = \sum_{k=1}^{n} \lambda_k y_k + \sum_{k=1}^{n} \lambda_k (x_k - y_k) \subseteq \operatorname{conv}(X) + B_\varepsilon.$$

Since $x \in \operatorname{conv}(A)$ was arbitrary we obtain

$$\operatorname{conv}(A) \subseteq \operatorname{conv}(X) + B_\varepsilon. \tag{4.3}$$

It is clear that $\operatorname{conv}(X)$ is compact because X is finite. Hence there is a set finite $Y \subseteq E$ such that $\operatorname{conv}(X) \subseteq Y + B_{\varepsilon_1 - \varepsilon}$. With (4.3) this implies

$$\operatorname{conv}(A) \subseteq \operatorname{conv}(X) + B_\varepsilon \subseteq Y + B_{\varepsilon_1 - \varepsilon} + B_\varepsilon \subseteq Y + B_{\varepsilon_1}. \qquad \square$$

4.5 Proposition. *Suppose that E is a normed space and $A \subseteq E$ precompact. Then* $\operatorname{conv}(A)$ *is precompact.*

Proof. Suppose that $\varepsilon > 0$. Since A is precompact, there is a finite set $X \subseteq E$ such that $A \subseteq X + B_{\varepsilon/2}$. By Lemma 4.4 there is a finite set $Y \subseteq E$ such that $\operatorname{conv}(A) \subseteq Y + B_\varepsilon$. $\square$

4.3 Compact Operators

4.6 Definition. Suppose that E, F are Banach spaces, $A \subseteq E$ and $f\colon A \to F$ is continuous. We say that f is *compact* if $f(A)$ is relatively compact. Denote by $\mathcal{K}(A, F)$ the set of continuous compact operators (maps) from A to F. f is *completely continuous* if f is continuous and if $f(B)$ is relatively compact for all bounded subsets B of A. f is of *finite dimension* if there is a subspace of finite dimension in F that contains $f(A)$. Denote the set of continuous maps of finite dimension from A to F by $\mathcal{F}(A,F)$.

Suppose that $A \subseteq E$ is bounded and closed and $f\colon A \to F$. f is called *proper* if $f^{-1}(K)$ is compact for every compact subset $K \subseteq F$.

4.7 Proposition. *Suppose that E, F are Banach spaces and $A \subseteq E$ is closed and bounded. Then*

(a) $\mathcal{F}(A,F) \cap \mathcal{K}(A,F)$ is dense in $\mathcal{K}(A,F)$ with respect to the supremum norm.

(b) If $f \in \mathcal{K}(A,E)$, then $I - f$ is proper.

Proof. **(a):** Suppose that $f \in \mathcal{K}(A,F)$ and $\varepsilon > 0$. There is a finite set $Y \subseteq F$ such that $\overline{f(A)} \subseteq Y + B_\varepsilon$, because $\overline{f(A)}$ is compact. Define maps $\varphi_y, \psi_y\colon \overline{f(A)} \to \mathbb{R}_0^+$ by

$$\varphi_y(x) := \max\{\,0,\ \varepsilon - |x-y|\,\} \qquad \text{for } y \in Y$$

and

$$\psi_y(x) := \frac{\varphi_y(x)}{\sum_{z\in Y}\varphi_z(x)}.$$

Clearly φ_y is continuous. Since $Y + B_\varepsilon$ covers $\overline{f(A)}$, the sum is positive in $\overline{f(A)}$. Since the sum is finite, ψ_y is continuous. For all $y \in Y$ and $x \in \overline{f(A)}$ we have $\psi_y(x) \in [0,1]$ and $\sum_{y\in Y}\psi_y(x) = 1$.

We define $g\colon A \to F$ by

$$g(x) := \sum_{y\in Y}\psi_y(f(x))y.$$

Clearly, g is continuous. g has its image in $[Y]$, the subspace of finite dimension generated by Y, that is $g \in \mathcal{F}(A,F)$. For $x \in A$ we obtain from the properties of ψ_y:

$$\|g(x) - f(x)\|_F = \Bigl\|\sum_{y\in Y}\psi_y(f(x))(y - f(x))\Bigr\|_F \le \sum_{y\in Y}\psi_y(f(x))\|y - f(x)\|_F \le \sum_{y\in Y}\psi_y(f(x))\varepsilon = \varepsilon.$$

It follows that $\sup_{x\in A}\|f(x) - g(x)\|_F \le \varepsilon$.

(b): Suppose that $K \subseteq F$ is compact, and suppose that $(x_n) \subseteq (I-f)^{-1}(K)$. Write $y_n := (I-f)(x_n) = x_n - f(x_n)$. Since $(y_n) \subseteq K$ and $(f(x_n))$ is relatively compact, passing to a subsequence we may assume that $y_n \to y$ and $f(x_n) \to z$ as $n \to \infty$. It follows that $x_n = y_n + f(x_n) \to y + z$ as $n \to \infty$. We conclude using that $(I-f)^{-1}(K)$ is closed. □

4.8 Proposition. *Suppose that E, F are Banach spaces, $A \subseteq E$ is closed and bounded, and suppose that $f \in \mathcal{K}(A,F)$. Then there is an extension $g \in \mathcal{K}(E,F)$ of f such that $g(E) \subseteq \operatorname{conv}(f(A))$.*

Proof. By Remark 4.3(c) relative compactness and precompactness coincide in F. We find the extension g such that $g(E) \subseteq \operatorname{conv}(A)$ with Tietze's Theorem, Theorem 2.9. Proposition 4.5 implies that $\operatorname{conv}(f(A))$ is relatively compact. Therefore, $g \in \mathcal{K}(E, F)$. □

4.9 Lemma. *Suppose that E, F are Banach spaces, $A \subseteq E$ is closed and bounded, and $f \colon A \to F$ continuous and proper. Then f is a closed map.*

Proof. Suppose that $B \subseteq A$ is closed, $(y_n) \subseteq f(B)$ such that $y_n \to y$ in F. There is $(x_n) \subseteq B$ such that $f(x_n) = y_n$. Since f is proper and $\{\, y_n \,\}_{n=1}^{\infty} \cup \{y\}$ compact, $\{\, x_n \,\}_{n=1}^{\infty}$ is relatively compact. Passing to a subsequence suppose that $x_n \to x$ in E. B closed implies that $x \in B$, that is, $f(x) \in f(B)$ and $y_n = f(x_n) \to f(x) = y$. Hence $y \in f(B)$. □

4.4 The Leray-Schauder Degree

4.10 Definition. Suppose that E is a Banach space. Denote by $\mathcal{A}_E$ the *set of admissible triplets* (f, Ω, y) where $\Omega \subseteq E$ is open and bounded, $f = I - K$, $K \in \mathcal{K}(\overline{\Omega}, E)$ and $y \in E \backslash f(\partial\Omega)$.

4.11 Remark. Definition 4.10 is consistent with the Definition of $\mathcal{A}_E$ for Banach spaces E of finite dimension given in Section 3.2, because in this case $I - f$ is always compact, that is, $f = I - K$ with compact K.

If $(f, \Omega, y) \in \mathcal{A}_E$, $f = I - K$ with $K \in \mathcal{K}(\overline{\Omega}, E)$, we set $\rho := \operatorname{dist}(y, f(\partial\Omega))$. By Proposition 4.7(b) f is proper, and therefore $f(\partial\Omega)$ is closed by Lemma 4.9. Hence $\rho > 0$. Proposition 4.7(a) provides us with $K_1 \in \mathcal{F}(\overline{\Omega}, E)$ such that $\sup_{x \in \overline{\Omega}} \|K(x) - K_1(x)\|_E < \rho$. If F is a subspace of E of finite dimension with $K_1(\overline{\Omega}) \subseteq F$ and $y \in F$, then we set $\Omega_F := \Omega \cap F$, $f_1 := (I - K_1)$. In consequence, $(f_1|_{\overline{\Omega_F}}, \Omega_F, y) \in \mathcal{A}_F$.

4.12 Lemma. *In the preceding situation* $\deg(f_1|_{\overline{\Omega_F}}, \Omega_F, y)$ *does not depend on K_1 nor F.*

Proof. Suppose that $K_i \in \mathcal{F}(\overline{\Omega}, E)$ is such that $\sup_{x \in \overline{\Omega}} \|K(x) - K_i(x)\|_E < \rho$, suppose that F_i are subspaces of E of finite dimension such that $K_i(\overline{\Omega}) \subseteq F_i$ and $y \in F_i$, and define $\Omega_i := \Omega \cap F_i$ and $f_i := (I - K_i)$, for $i = 1, 2$. Denote $F_0 := F_1 + F_2$ and $\Omega_0 := \Omega \cap F_0$. Proposition 3.11 yields

$$\deg(f_i|_{\overline{\Omega_0}}, \Omega_0, y) = \deg(f_i|_{\overline{\Omega_i}}, \Omega_i, y) \tag{4.4}$$

for $i = 1, 2$. Since $\partial\Omega_0 \subseteq \partial\Omega \cap F_0$, we have $\operatorname{dist}(y, f(\partial\Omega_0)) \geq \rho$. Put $h(t, x) := (1 - t) f_1(x) + t f_2(x)$. If $t \in [0, 1]$ and $x \in \partial\Omega_0$, then

$$\begin{aligned}
\|h(t, x) - y\| &= \|(1 - t)(f_1(x) - f(x)) + t(f_2(x) - f(x)) + f(x) - y\| \\
&\geq \|f(x) - y\| - \big((1 - t)\|f_1(x) - f(x)\| + t\|f_2(x) - f(x)\|\big) \\
&\geq \rho - \big((1 - t)\|K_1(x) - K(x)\| + t\|K_2(x) - K(x)\|\big) \\
&> 0.
\end{aligned}$$

In consequence, $y \notin h(t, \partial\Omega_0)$ for all $t \in [0,1]$. This implies that

$$\deg(f_1|_{\overline{\Omega_0}}, \Omega_0, y) = \deg(f_2|_{\overline{\Omega_0}}, \Omega_0, y), \tag{4.5}$$

and together with (4.4) we conclude. □

4.13 Definition. In the situation of Lemma 4.12 we define the *Leray-Schauder Degree* by

$$\deg(f, \Omega, y) := \deg(f_1|_{\overline{\Omega_F}}, \Omega_F, y). \tag{4.6}$$

Without proof we formulate the

4.14 Theorem. *For a Banach space E the map* $\deg\colon \mathcal{A}_E \to \mathbb{Z}$ *has the properties* (D1)–(D7)*, where h in* (D3) *has the form $h = I_E - H$, that is, $h(t,x) = x - H(t,x)$, and $H \in \mathcal{K}([0,1] \times \overline{\Omega}, E)$.*

4.15 Remark. One can prove that deg is the unique map $\mathcal{A}_E \to \mathbb{Z}$ with properties (D1)–(D3).

4.16 Theorem (Schauder)**.** *Suppose that E is a Banach space, $A \subseteq E$ is not empty, bounded, closed and convex, and $K\colon A \to A$ compact. Then K has a fixed point.*

Proof. By Proposition 4.8 we can extend K to E by $K_1 \in \mathcal{K}(E,E)$ such that $K_1(E) \subseteq A$. Selecting $r > 0$ such that $A \subseteq B_r E$, we define the continuous operator $H(t,x) := tK_1(x)$ for $(t,x) \in [0,1] \times \overline{B}_r E$. If $(t_n) \subseteq [0,1]$ and $(x_n) \subseteq \overline{B}_r$, then by the compactness of K_1 in $\overline{B}_r$ we may assume that (t_n) and $(K_1(x_n))$ converge, after passing to a subsequence. It follows that $(t_n K_1(x_n))$ converges. This implies that H is compact. For the homotopy $h := I - H$ it holds true that $0 \notin h(t, S_r E)$ if $t \in [0,1]$ because $tK_1(x) \neq x$ if $x \in S_r E$ and $t \in [0,1]$. Properties (D3) and (D1) yield

$$\deg(I - K_1, B_r, 0) = \deg(I, B_r, 0) = 1,$$

and hence that K_1 has a point fixed x_0 in $\overline{B}_r$. Clearly $x_0 \in A$, and therefore x_0 is a fixed point of K. □

4.17 Theorem (Schäfer)**.** *Suppose that E is a Banach space and suppose that $K\colon E \to E$ is completely continuous. Define*

$$\Lambda := \{\, x \in E \mid x = tK(x) \text{ for some } t \in [0,1] \,\}.$$

If Λ is bounded then K has a fixed point.

Proof. If Λ is bounded then there is $r > 0$ such that $\Lambda \subseteq B_r$. The restriction of K to $\overline{B}_r$ is a compact operator. As in the proof of Theorem 4.16 we obtain that $H\colon [0,1] \times \overline{B}_r \to E$, given by $H(t,x) := tK(x)$, is compact. If $t \in [0,1]$ then $x \neq tK(x)$ for $x \in S_r$ and therefore $0 \notin h(t, S_r)$, where $h := I_E - H$. In consequence

$$\deg(I - K, B_r, 0) = \deg(I, B_r, 0) = 1,$$

that is, K has a fixed point in B_r. □

4.18 Theorem. *Suppose that E is a Banach space, F a closed subspace of E, $\Omega \subseteq E$ is open and bounded, $K \in \mathcal{K}(\overline{\Omega}, F)$ and $y \in F$ is such that $(f, \Omega, y) \in \mathcal{A}_E$ for $f := I - K$. Set $\Omega' := \Omega \cap F$. Then $(f|_{\overline{\Omega'}}, \Omega', y) \in \mathcal{A}_F$ and*

$$\deg_E(f, \Omega, y) = \deg_F(f|_{\overline{\Omega'}}, \Omega', y).$$

Proof. Suppose that $K_1 \in \mathcal{F}(\overline{\Omega}, F)$ is an approximation such that $\|K - K_1\|_\infty < \operatorname{dist}(y, f(\partial\Omega))$. Suppose moreover that F_1 is a subspace of F of finite dimension such that it contains the image of $\overline{\Omega}$ under K_1 and y. Then

$$\deg(f, \Omega, y) = \deg(f|_{\overline{\Omega \cap F_1}}, \Omega \cap F_1, y) = \deg_F(f|_{\overline{\Omega'}}, \Omega', y)$$

by Proposition 3.11. □

4.5 The Degree of a Linear Operator

We calculate the degree of $I - K$ if K is a *linear* completely continuous operator and $I - K$ is a Banach space isomorphism. This case is important in applications.

4.19 Definition. A linear completely continuous operator is called a *compact linear operator*. Denote by $\mathcal{L}_c(E, F)$ the linear and completely continuous operators from E into F if E, F are normed spaces.

4.20 Reminder. Suppose that E is a real Banach space and $L \in \mathcal{L}(E)$ a bounded linear operator in E. The *resolvent set* $\rho(L)$ of L is the set of values $\lambda \in \mathbb{R}$ such that $\lambda I - L$ is bijective. By the open mapping theorem, $(\lambda I - L)^{-1} \in \mathcal{L}(E)$ in this case. The *spectrum* $\sigma(L)$ of L is the complement of $\rho(L)$ in $\mathbb{R}$. It always holds true that $\sigma(L)$ is compact and contained in $\overline{B}_{\|L\|}(0; E)$. A number $\lambda \in \mathbb{R}$ such that $\mathcal{N}(\lambda I - L) \neq \{0\}$ is an *eigenvalue* of L.

4.21 Proposition. *Suppose that E is a Banach space of infinite dimension and $K \in \mathcal{L}(E)$ compact. Then the following properties are satisfied:*

(a) *$\sigma(K)$ is at most countable and contains 0. If $\sigma(K)$ is infinite, then it consists of 0 and a sequence of eigenvalues that converges to 0.*

(b) *For all $\lambda \in \sigma(K) \setminus \{0\}$ there are unique closed subspaces $\mathcal{N}^*(\lambda)$ and $\mathcal{R}^*(\lambda)$ that are invariant under K such that $E = \mathcal{N}^*(\lambda) \oplus \mathcal{R}^*(\lambda)$, $\dim(\mathcal{N}^*(\lambda)) < \infty$, $\sigma(K|_{\mathcal{N}^*(\lambda)}) = \{\lambda\}$ and $\sigma(K|_{\mathcal{R}^*(\lambda)}) = \sigma(K) \setminus \{\lambda\}$. $\mathcal{N}^*(\lambda)$ is called the* generalized eigenspace of the eigenvalue λ. *The operator $(\lambda I - K) \colon \mathcal{R}^*(\lambda) \to \mathcal{R}^*(\lambda)$ is a Banach space isomorphism.*

(c) *If $\lambda, \mu \in \sigma(K) \setminus \{0\}$ are distinct, then $\mathcal{N}^*(\mu) \subseteq \mathcal{R}^*(\lambda)$.*

Sketch of proof. Item (a) is standard and can be found in texts on functional analysis. The idea for item (b) is to show, as in the finite dimensional case, that for $\lambda \in \sigma(K) \setminus \{0\}$ there are minimal $k(\lambda), \ell(\lambda) \in \mathbb{N}$ such that $\mathcal{N}((\lambda I - K)^{k(\lambda)}) = \mathcal{N}((\lambda I - K)^{k(\lambda)+1})$ and

$\mathcal{R}((\lambda I - K)^{\ell(\lambda)}) = \mathcal{R}((\lambda I - K)^{\ell(\lambda)+1})$. It then follows that $k(\lambda) = \ell(\lambda)$. One defines $\mathcal{N}^*(\lambda) := \mathcal{N}((\lambda I - K)^{k(\lambda)})$ and $\mathcal{R}^*(\lambda) := \mathcal{R}((\lambda I - K)^{k(\lambda)})$ and proves the stated properties.

To show (c) suppose that $\lambda, \mu \in \sigma(K)\backslash\{0\}$ are distinct and assume that $x \in \mathcal{N}^*(\mu)$. By the decomposition $E = \mathcal{N}^*(\lambda) \oplus \mathcal{R}^*(\lambda)$ there are unique elements $y \in \mathcal{N}^*(\lambda)$ and $z \in \mathcal{R}^*(\lambda)$ such that $x = y + z$. Set $L := (\mu I - K)^{k(\mu)}$. Then $0 = Lx = Ly + Lz$. Obviously, $\mathcal{N}^*(\lambda)$ and $\mathcal{R}^*(\lambda)$ are invariant under L. This implies that $Ly \in \mathcal{N}^*(\lambda)$ and $Lz \in \mathcal{R}^*(\lambda)$, that is, $Ly = Lz = 0$. Since λ is the unique eigenvalue of K in $\mathcal{N}^*(\lambda)$, L is injective in $\mathcal{N}^*(\lambda)$, and therefore $y = 0$. Hence $x = z \in \mathcal{R}^*(\lambda)$. □

4.22 Remark. In the preceding proposition, if $\lambda \in \sigma(K)\backslash\{0\}$, then $\dim \mathcal{N}(\lambda I - K)$ is called the *geometric multiplicity* and $\dim \mathcal{N}^*(\lambda)$ the *algebraic multiplicity* of λ.

4.23 Remark. If $K \in \mathcal{L}_c(E)$ and $I - K$ is injective, then 1 is not an eigenvalue of K. Since all elements of $\sigma(K)\backslash\{0\}$ are eigenvalues, $1 \notin \sigma(K)$. This implies that $I - K$ is a Banach space isomorphism.

4.24 Theorem. *Suppose that $K \in \mathcal{L}_c(E)$ is such that $I - K$ is injective. Let m denote the sum of the algebraic multiplicities of the eigenvalues $\lambda > 1$ of K (these coincide with the negative eigenvalues of $I - K$). Then*

$$\deg(I - K, B_1, 0) = (-1)^m.$$

Proof. By Proposition 4.21(a) there is only a finite number $\lambda_1, \lambda_2, \dots, \lambda_n$ of eigenvalues of K larger than 1. According to Proposition 4.21(b) we form the spaces $\mathcal{N}^*(\lambda_i)$ and $\mathcal{R}^*(\lambda_i)$ and we denote

$$\mathcal{N} := \bigoplus_{i=1}^n \mathcal{N}^*(\lambda_i) \qquad \text{and} \qquad \mathcal{R} := \bigcap_{i=1}^n \mathcal{R}^*(\lambda_i).$$

We need to show that

$$E = \mathcal{N} \oplus \mathcal{R}. \tag{4.7}$$

To prove $\mathcal{R} \cap \mathcal{N} = \{0\}$, suppose that $x \in \mathcal{R} \cap \mathcal{N}$. There are $x_i \in \mathcal{N}^*(\lambda_i)$, $a_i \in \mathbb{R}$, such that $x = \sum_{i=1}^n a_i x_i$. By Proposition 4.21(c) $\sum_{i \geq 2} a_i x_i \in \mathcal{R}^*(\lambda_1)$. Moreover, $x \in \mathcal{R} \subseteq \mathcal{R}^*(\lambda_1)$. Hence $a_1 x_1 = x - \sum_{i \geq 2} a_i x_i \in \mathcal{R}^*(\lambda_1) \cap \mathcal{N}^*(\lambda_1) = \{0\}$. Similarly one shows that $a_i x_i = 0$ for $i = 2, \dots, n$. It follows that $x = 0$. To see $E \subseteq \mathcal{N} + \mathcal{R}$, assume $x \in E$. There are $x_i \in \mathcal{N}^*(\lambda_i)$, $y_i \in \mathcal{R}^*(\lambda_i)$ such that $x = x_i + y_i$ for all $i = 1, 2, \dots, n$. Using Proposition 4.21(c) we obtain for every $k \in \{1, 2, \dots, n\}$:

$$x - \sum_{i=1}^n x_i = x - x_k - \sum_{i \neq k} x_i = y_k - \sum_{i \neq k} x_i \in \mathcal{R}^*(\lambda_k).$$

Hence $x - \sum_{i=1}^n x_i \in \mathcal{R}$. Since $\sum_{i=1}^n x_i \in \mathcal{N}$, this proves $x \in \mathcal{N} + \mathcal{R}$.

It is clear from Proposition 4.21 that $\mathcal{N}$ and $\mathcal{R}$ are invariant under K and that

$$\sigma(K|_{\mathcal{N}}) \subseteq (1, \infty) \qquad \text{and} \qquad \sigma(K|_{\mathcal{R}}) \subseteq (-\infty, 1). \tag{4.8}$$

Denote by P, Q the pair of projections corresponding to the splitting (4.7) and define the homotopy $h(t,\cdot) := I - K(P + (1-t)Q)$ from $I - K$ to $I - KP$. Clearly, the map $[0,1] \times B_1 \to E$, $(t,x) \mapsto K[Px + (1-t)Qx]$ is compact. If $t \in [0,1]$ and $x \in E$ satisfy $h(t,x) = 0$, then $Px + Qx = x = KPx + (1-t)KQx$. By the invariance of $\mathcal{N}$ and $\mathcal{R}$ under K, $KPx \in \mathcal{N}$ and $KQx \in \mathcal{R}$. Therefore, $KPx = Px$, which implies that $Px = 0$, since K does not have the eigenvalue 1. On the other hand, we have $(1-t)KQx = Qx$. If $t = 1$ then $Qx = 0$. If $t < 1$ then $1/(1-t) > 1$, and $KQx = \frac{1}{1-t}Qx$ implies $Qx = 0$ by (4.8). This proves $x = 0$ and yields that h is an admissible homotopy on B_1. Therefore, Theorem 4.18 implies

$$
\begin{aligned}
\deg(I - K, B_1, 0) &= \deg(I - KP, B_1, 0) \\
&= \deg((I - K)|_{\mathcal{N}}, B_1 \cap \mathcal{N}, 0) && \text{since } KP(E) \subseteq \mathcal{N} \\
&= (-1)^m && \text{by Proposition 2.22.}
\end{aligned}
$$

In the last equality we have used that $\dim \mathcal{N} = m$ and that $(I - K)|_{\mathcal{N}}$ has only negative eigenvalues. □

4.6 The Index of an Isolated Zero

It is often convenient to suppose that there is only a finite number of solutions and then use the additivity of the degree to show multiplicity of solutions. The following notion simplifies this argument:

4.25 Definition. Suppose that E is a Banach space, $x \in E$, $\Omega \subseteq E$ is a neighborhood of x and $K\colon \Omega \to E$ is completely continuous. Suppose moreover that x is the unique zero of $I - K$ in Ω. Then the *index of* $I - K$ *in* x (or the *fixed point index of* K *in* x) is the integer $\operatorname{ind}(K, x) := \lim_{r\to 0} \deg(I - K, B_r(x), 0)$. Note that by property (D2) $\deg(I - K, B_r(x), 0)$ is independent of $r > 0$ as long as r is small enough.

Since the index is a locally defined object, it can be calculated for differentiable maps by using their derivatives. To see this we need the

4.26 Lemma. *Suppose that E, F are Banach spaces, $\Omega \subseteq E$ is open and $K\colon \Omega \to F$ is completely continuous and Frechet differentiable at some $x \in \Omega$. Then $\mathrm{D}K(x) \in \mathcal{L}_{\mathrm{c}}(E, F)$.*

Proof. For $y \in E$ set $\omega(y) := K(x + y) - K(x) - \mathrm{D}K(x)y$. The differentiability of K at x implies that $\omega(y)/\|y\| \to 0$ as $\|y\| \to 0$.

It suffices to show that $\mathrm{D}K(x)B_1$ is precompact. Let $\varepsilon > 0$ and pick $\delta > 0$ such that $x + B_\delta \subseteq \Omega$ and $\omega(y)/\|y\| \le \varepsilon$ for $y \in B_\delta$. Since $K(x + B_\delta)$ is compact, there is a finite set $X \subseteq E$ such that $-K(x) + K(x + B_\delta) \subseteq X + B_{\varepsilon\delta}$. For $y \in B_\delta$ we have

$$\mathrm{D}K(x)y = -K(x) + K(x + y) - \omega(y)$$

and $\|\omega(y)\| \le \varepsilon\delta$. Hence

$$\delta \mathrm{D}K(x)B_1 = \mathrm{D}K(x)B_\delta \subseteq -K(x) + K(x + B_\delta) + B_{\varepsilon\delta} \subseteq X + B_{2\varepsilon\delta}$$

and therefore $\mathrm{D}K(x)B_1 \subseteq \frac{1}{\delta}X + B_{2\varepsilon}$. Since $\varepsilon > 0$ was arbitrary, $\mathrm{D}K(x)B_1$ is precompact. □

4.27 Proposition. *In the situation of Definition 4.25 suppose that K is Frechet differentiable in x such that $I - \mathrm{D}K(x)$ is injective. Then*

$$\operatorname{ind}(K, x) = \deg(I - \mathrm{D}K(x), B_1, 0).$$

In particular, $|\operatorname{ind}(K, x)| = 1$.

Proof. Without loss of generality we may assume that $x = 0$ and therefore $K(0) = 0$. Define $H(t, x) := (1 - t)K(x) + t\mathrm{D}K(0)x$ for $(t, x) \in [0, 1] \times \Omega$. Then H is completely continuous. We will show that $I_E - H$ is admissible in a sufficiently small ball B_r. If this were not true there would exist sequences $(t_n) \subseteq [0, 1]$ and $x_n \to 0$ in E such that $H(t_n, x_n) = x_n$ for all n. It would follow that

$$(I_E - \mathrm{D}K(0))\frac{x_n}{\|x_n\|} = \frac{(1 - t_n)(K(x_n) - \mathrm{D}K(0)x_n)}{\|x_n\|} \to 0 \qquad \text{as } n \to \infty.$$

This would imply that $0 \in \sigma(I_E - \mathrm{D}K(0))$ and hence that 0 is an eigenvalue of $I_E - \mathrm{D}K(0)$, by the compactness of $\mathrm{D}K(0)$, in contradiction with the injectivity of $I_E - \mathrm{D}K(0)$. Therefore, there is $r_0 > 0$ such that for every $r \in (0, r_0]$ it holds true that $\deg(I_E - K, B_r, 0) = \deg(I_E - \mathrm{D}K(0), B_r, 0) = \deg(I_E - \mathrm{D}K(0), B_1, 0)$. ☐

4.28 Remark. Without supposing the injectiveness of the derivative the preceding proposition is not true. To see this, consider $E := \mathbb{R}$ and $K(x) := x - x^2$. Then 0 is the unique fixed point of K but $\operatorname{ind}(K, 0) = \deg(\mathrm{id} - K, B_1, 0) = 0$ since the function $(\mathrm{id} - K)(x) = x^2$ is positive in -1 and 1. Here $(\mathrm{id} - \mathrm{D}K(0))(x) = x - \mathrm{D}K(0)x \equiv 0$, that is, $\mathrm{id} - \mathrm{D}K(0)$ is not injective.

5 Applications to Partial Differential Equations

5.1 Boundary Value Problems

In this section let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with smooth boundary. We are interested in *strong solutions* $u \in H^2(\Omega) \cap H_0^1(\Omega)$ of the elliptic problem

$$(5.1) \qquad -\Delta u(x) - \lambda u(x) = f(u(x)), \qquad \text{for almost every } x \in \Omega,$$

where $\lambda \in \mathbb{R}$ and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is locally Hölder continuous. By regularity theory, any strong solution of (5.1) is also a *classical solution*, i.e. a function in the space $C^2(\Omega) \cap C(\overline{\Omega})$ that satisfies

$$(5.2) \qquad \begin{cases} -\Delta u(x) - \lambda u(x) = f(u(x)), & \text{for all } x \in \Omega, \\ u(x) = 0, & \text{for all } x \in \partial\Omega. \end{cases}$$

The linear problem related to (5.1) is

$$(5.3) \qquad -\Delta u(x) - \lambda u(x) = g(x), \qquad \text{for almost every } x \in \Omega.$$

There is a set $\sigma(-\Delta) \subseteq \mathbb{R}$ (the *spectrum of* $-\Delta$ *with respect to homogeneous Dirichlet conditions on the boundary*) such that if $\lambda \in \mathbb{R} \backslash \sigma(-\Delta)$, then for all $g \in L^2(\Omega)$ there is a unique function $u = Lg \in H^2(\Omega) \cap H_0^1(\Omega)$ that is a solution of (5.3). Moreover,

$$(5.4) \qquad L = (-\Delta - \lambda)^{-1} \in \mathcal{L}(L^2(\Omega), H^2(\Omega) \cap H_0^1(\Omega)).$$

Since the embedding $H^2(\Omega) \hookrightarrow L^2(\Omega)$ is compact, also the linear operator $L\colon L^2(\Omega) \to L^2(\Omega)$ is compact.

5.2 An Application of *A Priori* Bounds

5.1 Theorem. *Suppose that* $\lambda \in \mathbb{R} \backslash \sigma(-\Delta)$, *and that* $f\colon \mathbb{R} \to \mathbb{R}$ *is Lipschitz continuous and satisfies* $\lim_{|s|\to\infty} f(x,s)/s = 0$. *Then Equation* (5.2) *has a classical solution.*

Proof. For $u \in L^2$ we define $F(u)\colon \overline{\Omega} \to \mathbb{R}$ by

$$F(u)(x) := f(u(x)).$$

Denote by C_1 a Lipschitz constant for f. For any $u, v \in L^2$ it follows that $F(u), F(v)$ are measurable. Moreover,

$$|F(u)(x)| \le |f(u(x)) - f(0)| + |f(0)| \le C_1|u(x)| + |f(0)|$$

and

$$|F(u)(x) - F(v)(x)| = |f(u(x)) - f(v(x))| \le C_1|u(x) - v(x)|.$$

Therefore, $F(u) \in L^2$, $|F(u)|_2 \le |f(0)||\Omega|^{1/2} + C_1|u|_2$ and $|F(u) - F(v)|_2 \le C_1|u - v|_2$. We conclude that F is continuous and maps bounded subsets of L^2 into bounded subsets of L^2.

As before we set $L := (-\Delta - \lambda)^{-1}$. There is $C > 0$ such that

$$|f(s)| \le C + \frac{|s|}{2\|L\|} \qquad \text{for all } s \in \mathbb{R}.$$

This implies the existence of (another) $C > 0$ such that

$$|F(u)|_2 \le C + \frac{|u|_2}{2\|L\|} \qquad \text{for all } u \in L^2. \tag{5.5}$$

Finding a solution of (5.1) is equivalent to finding a fixed point of $K := L \circ F$:

$$(-\Delta + \lambda)u = F(u) \Leftrightarrow u = LF(u) = K(u).$$

We will apply Schäfer's Theorem, Theorem 4.17: First we note that K is absolutely continuous by the properties of F and since L is compact as an operator $L^2 \to L^2$. Suppose that $u \in L^2$ satisfies $u = tK(u)$ for some $t \in [0, 1]$. Then by (5.5)

$$|u|_2 \le \|K(u)\| \le \|L\||F(u)|_2 \le C\|L\| + \frac{1}{2}|u|_2,$$

i.e. $|u|_2 \le 2C\|L\|$. This is the necessary *a priori* bound on solutions u of $u = tK(u)$. Now Schäfer's Theorem implies the existence of a fixed point for K. □

5.3 An Exact Multiplicity Result

Denote by $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots$ the eigenvalues of $-\Delta$ with Dirichlet boundary conditions on the bounded smooth domain $\Omega \subseteq \mathbb{R}^N$.

5.2 Theorem. *Consider $f \in C^2(\mathbb{R})$ with the properties $f(0) = 0$, $f''(t)t > 0$ for $t \ne 0$, $\lim_{t\to\pm\infty} f'(t) = b_\pm$ and*

$$\lambda_{k-1} < f'(0) < \lambda_k < b_\pm < \lambda_{k+1}. \tag{5.6}$$

Then

$$\begin{cases} -\Delta u = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.7}$$

has exactly three strong solutions: the trivial solution $u \equiv 0$ and two nontrivial solutions.

Proof. Set $L := (-\Delta)^{-1}$ and $K(u) := LF(u)$ as before. Since f' ist strictly increasing on $[0,\infty)$ and strictly decreasing on $(-\infty,0]$, we have $f'(t) \in [f'(0),\lambda_{k+1})$ for all t. Since $f'(0) > 0$, it follows that $|f(t)| \le \lambda_{k+1}|t|$ for all t and $F \in C^1(L^2,L^2)$ with $(\mathrm{D}F(u)v)(x) = f'(u(x))v(x)$. Hence $K \in C^1(L^2,H^2)$ with $\mathrm{D}K = L \circ \mathrm{D}F$. Since $L \in \mathcal{L}(L^2,H^2)$, $L\colon L^2 \to H_0^1$ is a compact linear operator. Therefore, $K\colon L^2 \to H_0^1$ is completely continuous.

For any bounded and measurable function h on Ω we denote by $\mu_i(h)$ the i-th positive eigenvalue of the compact linear operator $v \mapsto L[hv]$, counted in decreasing order. By the variational characterization of eigenvalues we have

$$\mu_i(h) = \sup_{\substack{\Lambda \le H_0^1 \\ \dim\Lambda = i}} \min_{v\in\Lambda\setminus\{0\}} \frac{\langle Lhv, v\rangle}{\|v\|^2} = \sup_{\substack{\Lambda \le H_0^1 \\ \dim\Lambda = i}} \min_{v\in\Lambda\setminus\{0\}} \frac{\int_\Omega hv^2}{\|v\|^2}. \tag{5.8}$$

Here we use the norm $\|\cdot\|$ in H_0^1 given by $\|v\|^2 = \int_\Omega |\nabla v|^2$ and the associated scalar product $\langle\cdot,\cdot\rangle$. It follows easily that if $h_1 \le h_2$ on Ω and $h_1 < h_2$ on a set of positive measure, then $\mu_i(h_1) < \mu_i(h_2)$ for all $i \in \mathbb{N}$. For $h \equiv \lambda_i$ we find

$$\mu_i(\lambda_i) = \lambda_i \cdot \sup_{\substack{\Lambda \le H_0^1 \\ \dim\Lambda = i}} \min_{v\in\Lambda\setminus\{0\}} \frac{\int_\Omega v^2}{\|v\|^2} = \lambda_i \cdot \frac{1}{\lambda_i} = 1. \tag{5.9}$$

Note that $\mu_i(f'(u))$ is the i-th eigenvalue of $\mathrm{D}K(u)$.

We consider fixed points of K and claim that if $K(u) = u$, then u is isolated as a fixed point of K and

$$\operatorname{ind}(K,u) = \begin{cases} (-1)^{k-1}, & u = 0, \\ (-1)^k, & u \ne 0. \end{cases} \tag{5.10}$$

If $u = 0$ then $\mu_k(f'(0)) < \mu_k(\lambda_k) = 1$ and $\mu_{k-1}(f'(0)) > \mu_{k-1}(\lambda_{k-1}) = 1$. Hence 1 is not an eigenvalue of $\mathrm{D}K(0)$ and there are precisely $k-1$ eigenvalues larger than 1 for $\mathrm{D}K(0)$. If $u \ne 0$ then from $-\Delta u = f(u) = (f(u)/u)u$ we obtain $j \in \mathbb{N}$ such that $\mu_j(f(u)/u) = 1$. Observe that $f(t) = \int_0^t f'$ implies

$$\lambda_{k-1} < \frac{f(t)}{t} < \lambda_{k+1} \qquad \text{for all } t \in \mathbb{R} \tag{5.11}$$

and

$$\frac{f(t)}{t} < f'(t) \qquad \text{for } t \ne 0. \tag{5.12}$$

It follows from (5.11) that $\mu_{k+1}(f(u)/u) < \mu_{k+1}(\lambda_{k+1}) = 1$ and $\mu_{k-1}(f(u)/u) > \mu_{k-1}(\lambda_{k-1}) = 1$. This yields $j = k$. Since $u \ne 0$ on a set of positive measure we infer from (5.12) that $\mu_k(f'(u)) > \mu_k(f(u)/u) = 1$ and from the hypotheses that $\mu_{k+1}(f'(u)) < \mu_{k+1}(\lambda_{k+1}) = 1$. Hence 1 is not an eigenvalue of $\mathrm{D}K(u)$ and there are precisely k eigenvalues larger than 1 for $\mathrm{D}K(u)$. In both cases $I - \mathrm{D}K(u)$ is an isomorphism. The implicit function theorem implies that u is an isolated zero of $I - K$. By Proposition 4.27 (5.10) holds true.

Now we prove that there is $R > 0$ such that $|u|_2 < R$ for all solutions of (5.7) and

$$\deg(I - K, B_R L^2, 0) = (-1)^k. \tag{5.13}$$

Pick $b \in (\lambda_k, \lambda_{k+1})$ and define the completely continuous operator $H(t, u) := (1-t)K(u)+ tbu$. We prove that $I - H$ is admissible on B_R for the value 0 if R is large enough. If this is false then there exist sequences $(t_i) \subseteq [0, 1]$ and $u_i \in L^2$ such that $|u|_2 \to \infty$ and $H(t_i, u_i) = u_i$. Define $w_i := u_i/|u_i|_2$. It follows that

$$L\left[\left((1 - t_i)\frac{f(u_i)}{u_i} + t_i b\right) w_i\right] = w_i$$

for all i. Since $L \in \mathcal{L}_c(L^2, H_0^1)$, we may assume that $t_i \to t^*$ and $w_i \to w^*$ in H^1 and pointwise almost everywhere. Hence

$$L\big[\underbrace{((1 - t^*)\psi + t^* b)}_{h} w^*\big] = w^*$$

with

$$\psi(x) = \begin{cases} b_+ & w^*(x) > 0 \\ b_- & w^*(x) < 0 \\ b & w^*(x) = 0. \end{cases}$$

There is $j \in \mathbb{N}$ with $\mu_j(h) = 1$. On the other hand, $\lambda_k < h < \lambda_{k+1}$ implies $\mu_{k+1}(h) < \mu_{k+1}(\lambda_{k+1}) = 1$ and $\mu_k(h) > \mu_k(\lambda_k) = 1$, a contradiction. In particular, there is no sequence of solutions of (5.7) with L^2-norm tending to ∞. Hence all solutions of (5.7) are contained in B_R for some fixed large R and $I - H$ is admissible on B_R for the value 0. From $\mu_{k+1}(b) < \mu_{k+1}(\lambda_{k+1}) = 1$ and $\mu_k(b) > \mu_k(\lambda_k) = 1$ it follows that $\deg(I - b, B_R, 0) = (-1)^k$ and hence (5.13).

To conclude we recall that since $I - K$ is proper on B_R and all zeros are isolated, the set of zeros of $I - K$ is finite. Denote by m the number of nontrivial solutions. The excision property (D2), (5.10) and (5.13) imply

$$(-1)^k = (-1)^{k-1} + m(-1)^k,$$

that is, $m = 2$. □

Bibliography

[1] A. Ambrosetti and D. Arcoya, *An introduction to nonlinear functional analysis and elliptic problems*, Progress in Nonlinear Differential Equations and their Applications, vol. 82, Birkhäuser Boston, Inc., Boston, MA, 2011. MR 2816471

[2] A. Ambrosetti and A. Malchiodi, *Nonlinear analysis and semilinear elliptic problems*, Cambridge Studies in Advanced Mathematics, vol. 104, Cambridge University Press, Cambridge, 2007. MR 2292344

[3] C. Bandle and W. Reichel, *Solutions of quasilinear second-order elliptic boundary value problems via degree theory*, Stationary partial differential equations. Vol. I, Handb. Differ. Equ., North-Holland, Amsterdam, 2004, pp. 1–70. MR MR2103687 (2005m:35081)

[4] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin, 1985. MR MR787404 (86j:47001)

[5] P. Drábek and J. Milota, *Methods of nonlinear analysis*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 2007, Applications to differential equations. MR 2323436

[6] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224, Springer-Verlag, Berlin, 1983. MR MR737190 (86c:35035)

[7] A. Granas and J. Dugundji, *Fixed point theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003. MR 1987179 (2004d:58012)

[8] M.A. Krasnosel′skiĭ and P.P. Zabreĭko, *Geometrical methods of nonlinear analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 263, Springer-Verlag, Berlin, 1984, Translated from the Russian by Christian C. Fenske. MR MR736839 (85b:47057)

[9] D. Motreanu, V.V. Motreanu, and N. Papageorgiou, *Topological and variational methods with applications to nonlinear boundary value problems*, Springer, New York, 2014. MR 3136201

[10] D. O'Regan and R. Precup, *Theorems of Leray-Schauder type and applications*, Series in Mathematical Analysis and Applications, vol. 3, Gordon and Breach Science Publishers, Amsterdam, 2001. MR 1937722

[11] G. Vidossich, *An invitation to multiplicity results for elliptic problems*, Evolution equations and their applications (Schloss Retzhof, 1981), Res. Notes in Math., vol. 68, Pitman, Boston, Mass.-London, 1982, pp. 247–258. MR 668391

[12] E. Zeidler, *Nonlinear functional analysis and its applications. I*, Springer-Verlag, New York, 1986, Fixed-point theorems, Translated from the German by Peter R. Wadsack. MR 816732 (87f:47083)

[13] Z. Zhang, *Variational, topological, and partial order methods with their applications*, Developments in Mathematics, vol. 29, Springer, Heidelberg, 2013. MR 2964004