

Uniform Continuity and Brézis-Lieb Type Splitting for Superposition Operators in Sobolev Space

Nils Ackermann*

Instituto de Matemáticas

Universidad Nacional Autónoma de México

Circuito Exterior, C.U., 04510 México D.F., México

We prove a variant of the Brézis-Lieb Lemma that applies to more general nonlinear superposition operators within a certain range of growth exponents, at the expense of stronger conditions on the admissible sequences of functions. This new set of conditions is well adapted to second order semilinear elliptic partial differential equations on \mathbb{R}^N . The proof rests on the uniform continuity of superposition operators on bounded subsets of Sobolev space, which we obtain from an application of the concentration compactness method.

1 Introduction

In their seminal paper [4] Brézis and Lieb prove a result about the decoupling of certain integral expressions, which has been used as one route to concentration compactness in the calculus of variations. To describe a special case of this theorem, suppose that Ω is a domain in \mathbb{R}^N , $p > 1$, $f(t) := |t|^p$ for $t \in \mathbb{R}$, and (u_n) a bounded sequence in $L^p(\Omega)$ that converges pointwise almost everywhere to some function u . If one denotes by $\mathcal{F}: L^p(\Omega) \rightarrow L^1(\Omega)$ the superposition operator f generates, i.e., $\mathcal{F}(v)(x) := f(v(x))$, then $u \in L^p(\Omega)$ and

$$\mathcal{F}(u_n) - \mathcal{F}(u_n - u) \rightarrow \mathcal{F}(u) \quad \text{in } L^1(\Omega), \text{ as } n \rightarrow \infty. \quad (1.1)$$

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The same conclusion is obtained in that paper for more general functions f , imposing conditions on f that are satisfied for continuous convex functions with $f(0) = 0$, and imposing additional conditions on the sequence (u_n) .

Our aim is to give a similar result under a different set of hypotheses that, on the one hand, applies to a larger class of functions f within a certain range of exponents p , but, on the other hand, restricts the admissible sequences. This choice of hypotheses is justified by the numerous applications where they are satisfied.

Using the standard notation $2^* := \infty$ if $N = 1$ or $N = 2$, and $2^* := 2N/(N-2)$ if $N \geq 3$, recall the continuous embedding of the Sobolev space $H^1(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$ for $p \in [2, 2^*)$. In our main result we prove (1.1) for a continuous function f that is polynomially bounded with exponent $p \in (2, 2^*)$, and for sequences (u_n) that converge weakly in $H^1(\mathbb{R}^N)$. We also consider \mathcal{F} taking values in more general Lebesgue spaces $L^\nu(\mathbb{R}^N)$.

In the proof we first need to show that \mathcal{F} is uniformly continuous on bounded subsets of $H^1(\mathbb{R}^N)$, with respect to the L^p -norm (and hence also with respect to the H^1 -norm), a result of independent interest. The restriction $p > 2$ appears since the proof rests on Lions' Vanishing Lemma. The uniform continuity of such \mathcal{F} on bounded subsets of $H^1(\mathbb{R}^N)$ is known as folklore in some circles, but we are not aware of a published proof of this nontrivial fact.

The restriction to $\Omega = \mathbb{R}^N$ is not accidental; the proofs require the existence of a group action by translation on Ω . We only strive for generality with respect to the conditions on f , but confine ourselves to a simple functional setup to highlight the main idea. Our result could also be proved for more general domains Ω (always assuming an adequate translation action) and other spaces than H^1 .

We do allow explicit dependence of f on x and need to introduce some terminology. The function $f: \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is a *Caratheodory function* if f is measurable and if $f(x, \cdot)$ is continuous for almost every $x \in \mathbb{R}^N$. The induced superposition operator on functions $u: \mathbb{R}^N \rightarrow \mathbb{R}$ is then given by $\mathcal{F}(u)(x) := f(x, u(x))$. If A is a real invertible $N \times N$ -matrix then f is said to be A -periodic in its first argument if $f(x + Ak, t) = f(x, t)$ for all $x \in \mathbb{R}^N$, $k \in \mathbb{Z}^N$, and $t \in \mathbb{R}$. A map $\mathcal{F}: X \rightarrow Y$, where X is a Hilbert space and Y a Banach space, will be called *BL-splitting* (BL for Brézis-Lieb) if $\mathcal{F}(u_n) - \mathcal{F}(u_n - u) \rightarrow \mathcal{F}(u)$ in Y whenever $u_n \rightharpoonup u$ in X .

With these preparations our main result reads:

Theorem 1.1. *Consider $\mu > 0$, $\nu \geq 1$, and $C_0 > 0$, such that $p := \mu\nu \in (2, 2^*)$. Suppose that $f: \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caratheodory function that satisfies*

$$|f(x, t)| \leq C_0 |t|^\mu \quad \text{for all } x \in \mathbb{R}^N, t \in \mathbb{R}, \quad (1.2)$$

and which is A -periodic in its first argument, for some invertible matrix $A \in \mathbb{R}^{N \times N}$. Denote by $\mathcal{F}: L^p(\mathbb{R}^N) \rightarrow L^\nu(\mathbb{R}^N)$ the continuous superposition operator induced by f . Then \mathcal{F} is uniformly continuous on bounded subsets of $H^1(\mathbb{R}^N)$ with respect to the L^p - L^ν -norms and hence also with respect to the H^1 - L^ν -norms. Moreover, $\mathcal{F}: H^1(\mathbb{R}^N) \rightarrow L^\nu(\mathbb{R}^N)$ BL-splits.

Remark 1.2. The result also holds in a slightly restricted sense for functions f that are sums of functions as in Theorem 1.1, i.e., functions that satisfy merely

$$|f(x, t)| \leq C_0(|t|^{\mu_1} + |t|^{\mu_2}) \quad \text{for all } x \in \mathbb{R}^N, t \in \mathbb{R},$$

where $\mu_i \nu \in (2, 2^*)$ for $i = 1, 2$. In that case, $\mathcal{F}: H^1(\mathbb{R}^N) \rightarrow L^\nu(\mathbb{R}^N)$ is uniformly continuous on bounded subsets of $H^1(\mathbb{R}^N)$ with respect to the H^1 - L^ν norms, and \mathcal{F} BL-splits.

Theorem 1.1 is relevant in the calculus of variations, applied to the question of existence of solutions of certain second order semilinear elliptic partial differential equations in unbounded domains. It allows to apply the locally compact variant of the method of concentration compactness under weaker assumptions on the nonlinearity than known before and provides a convenient framework in many situations to avoid cumbersome cut-off arguments.

To give just one immediate improvement of a known result, consider the equation

$$-\Delta u + V(x)u = f(x, u) \quad u \in H^1(\mathbb{R}^N). \quad (1.3)$$

We assume that f and V are 1-periodic in all coordinates of x , V is continuous, $V > 0$, $f(x, \cdot)$ is continuously differentiable for almost all x , and $\partial_2 f$ is a Caratheodory function. We assume the standard Ambrosetti-Rabinowitz condition for f and $\partial_2 f(x, t)t^2 > f(x, t)t$ for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Suppose that there are $2 < p_1 \leq p_2 < 2^*$ such that

$$|\partial_2^k f(x, t)| \leq C(|t|^{p_1-k-1} + |t|^{p_2-k-1}) \quad (1.4)$$

for all $x \in \mathbb{R}^N$, $t \in \mathbb{R}$ and $k = 0, 1$. Then by Theorem 1.1 the corresponding results in [2] on the existence of multibump solutions of (1.3) hold true. In that paper we imposed (1.4) also for $k = 2$, assuming that f is twice differentiable in t . We could have done there with an appropriate growth bound on the Hölder constant for $\partial_2 f$. Theorem 1.1 removes the need for regularity conditions on $\partial_2 f$.

Our proof of Theorem 1.1 uses a combination of ideas from [1, 6–8]. It is very similar to the proof of [8, Theorem 3.1] but involves an intermediate cut-off step that yields, in the first iteration, a weak form of BL-splitting for \mathcal{F} , as in [2, Lemma 3.2]. The other ingredients are the local compactness of the Sobolev embedding $H^1 \hookrightarrow L^p$ for $p \in [2, 2^*)$ and Lions' Vanishing Lemma. We present the proof in an almost self contained form to make it more accessible.

2 Proof of the Theorem

For simplicity we will only consider the case $A = I$ (the identity transformation). Denote the respective translation action of the additive group \mathbb{Z}^N on functions $u: \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$(a \star u)(x) := u(x - a), \quad a \in \mathbb{Z}^N, x \in \mathbb{R}^N.$$

Let B_R denote, for $R > 0$, the open ball in \mathbb{R}^N with center 0 and radius R . If $r \in [1, \infty]$ and if (Ω, Σ, μ) is a measure space with a positive measure μ then denote by $|\cdot|_r$ the norm of $L^r(\mu)$. We omit \mathbb{R}^N in the notation for function spaces. Let $\langle \cdot, \cdot \rangle$ denote the standard scalar product in H^1 , defined by

$$\langle u, v \rangle := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv),$$

and let $\|\cdot\|$ denote the associated norm. Also denote by w-lim the weak limit of a weakly convergent sequence.

We first recall a functional consequence of Lions' Vanishing Lemma, [6, Lemma I.1.].

Lemma 2.1. *Suppose for a sequence $(u_n) \subseteq H^1$ that $a_n \star u_n \rightharpoonup 0$ in H^1 for every sequence $(a_n) \subseteq \mathbb{Z}^N$. Then $u_n \rightarrow 0$ in L^p for all $p \in (2, 2^*)$.*

Proof. Note first that (u_n) is bounded in H^1 since $u_n \rightharpoonup 0$ in H^1 . We claim that

$$\sup_{y \in \mathbb{R}^N} \int_{y+B_1} |u_n|^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.1)$$

If the claim were not true there would exist $\varepsilon > 0$ and a sequence $(y_n) \subseteq \mathbb{R}^N$ such that, after passing to a subsequence of (u_n) ,

$$\int_{y_n+B_1} |u_n|^2 \geq \varepsilon.$$

Pick $(a_n) \subseteq \mathbb{Z}^N$ such that $|a_n + y_n|_\infty < 1$ for all n . With $R := \sqrt{N} + 1$ it follows that $a_n + y_n + B_1 \subseteq B_R$ and hence

$$\int_{B_R} |a_n \star u_n|^2 \geq \varepsilon$$

for all n . We reach a contradiction since $a_n \star u_n \rightharpoonup 0$ in H^1 and hence $a_n \star u_n \rightarrow 0$ in $L^2(B_R)$ by the theorem of Rellich and Kondrakov. Therefore (2.1) holds.

The claim of the theorem now follows from [6, Lemma I.1.] with $p = q = 2$. Compare also with [8, Lemma 3.3]. \square

Proof of Theorem 1.1. The continuity of $\mathcal{F}: L^p \rightarrow L^\nu$ follows from (1.2) and the theory of superposition operators, see [3].

We start by proving the uniform continuity. Let $(u_{i,n}^0)_{n \in \mathbb{N}_0}$ be bounded sequences in H^1 for $i = 1, 2$ and set $C_1 := \max_{i=1,2} \limsup_{n \rightarrow \infty} \|u_{i,n}^0\|$. Suppose for a contradiction that

$$|u_{1,n}^0 - u_{2,n}^0|_p \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (2.2)$$

and that there is $C_2 > 0$ such that

$$|\mathcal{F}(u_{1,n}^0) - \mathcal{F}(u_{2,n}^0)|_\nu \geq C_2 \quad \text{for all } n. \quad (2.3)$$

Successively we will define a countable infinity, indexed by $k \in \mathbb{N}_0$, of sequences $(a_n^k)_n \subseteq \mathbb{Z}^N$ and $(u_{i,n}^k)_n \subseteq H^1$, $i = 1, 2$, with the following properties (among others, to be seen below):

$$\max_{i=1,2} \limsup_{n \rightarrow \infty} \|u_{i,n}^k\| \leq C_1, \quad (2.4)$$

$$\lim_{n \rightarrow \infty} |u_{1,n}^k - u_{2,n}^k|_p = 0, \quad (2.5)$$

$$\liminf_{n \rightarrow \infty} |\mathcal{F}(u_{1,n}^k) - \mathcal{F}(u_{2,n}^k)|_\nu \geq C_2, \quad (2.6)$$

and

$$\text{w-lim}_{n \rightarrow \infty} (-a_n^\ell) \star u_{i,n}^k = 0 \quad \text{in } H^1, \text{ if } \ell < k, \text{ for } i = 1, 2. \quad (2.7)$$

We need to say something about the extraction of subsequences. In order to obtain $(a_n^k)_n$ and $(u_{i,n}^{k+1})_n$ from $(u_{i,n}^k)_n$, we first pass to a subsequence of $(u_{i,n}^k)_n$ and then use its terms in the construction. Once the new sequences are built we may remove a finite number of terms at their start, with the goal of obtaining additional properties. Beginning with the following iteration there are no more retrospective changes to the sequences already built. The act of passing to subsequences will be implicit, leaving it to the reader to complete the argument, if so desired.

For $k = 0$ the properties (2.4)–(2.7) are fulfilled by the definition of C_1 and by (2.2) and (2.3). Assume now that (2.4)–(2.7) hold for some $k \in \mathbb{N}_0$. Denote by W_k the set of $v \in H^1$ such that there are a sequence $(a_n) \subseteq \mathbb{Z}^N$ and a subsequence of $(u_{1,n}^k)$ with $\text{w-lim}_{n \rightarrow \infty} a_n \star u_{1,n}^k = v$ in H^1 .

If $\text{w-lim}_{n \rightarrow \infty} a_n \star u_{1,n}^k = 0$ in H^1 were true for all sequences $(a_n) \subseteq \mathbb{Z}^N$, by Lemma 2.1 it would follow that $\lim_{n \rightarrow \infty} u_{1,n}^k = 0$ in L^p . Equation (2.5) and the continuity of \mathcal{F} on L^p would lead to a contradiction with (2.6). Therefore

$$q_k := \sup_{v \in W_k} \|v\| \in (0, C_1].$$

Pick $v^k \in W_k$ such that

$$\|v^k\| \geq \frac{q_k}{2} > 0. \quad (2.8)$$

There is $(a_n^k)_n \subseteq \mathbb{Z}^N$ such that $\text{w-lim}_{n \rightarrow \infty} (-a_n^k) \star u_{1,n}^k = v^k$ in H^1 . Since $\lim_{n \rightarrow \infty} (-a_n^k) \star u_{1,n}^k = v^k$ in L_{loc}^p by the Rellich-Kondrakov theorem, we may assume by (2.5) that

$$\text{w-lim}_{n \rightarrow \infty} (-a_n^k) \star u_{i,n}^k = v^k \quad \text{in } H^1, \text{ for } i = 1, 2, \quad (2.9)$$

i.e., the same property for both indices $i = 1, 2$. Let us write $z_n^k := (-a_n^k) \star u_{1,n}^k$. For $n \in \mathbb{N}_0$ define $Q_n: [0, \infty) \rightarrow [0, \infty)$ by

$$Q_n(R) := \int_{B_R} |z_n^k|^p.$$

The functions Q_n are uniformly bounded and nondecreasing. We may assume that (Q_n) converges pointwise almost everywhere to a bounded nondecreasing function Q [5]. It is easy to build a sequence $R_n \rightarrow \infty$ such that for every $\varepsilon > 0$ there is $R > 0$, arbitrarily large, with

$$\limsup_{n \rightarrow \infty} (Q_n(R_n) - Q_n(R)) \leq \varepsilon.$$

Hence

$$\forall \varepsilon > 0 \exists R > 0: \limsup_{n \rightarrow \infty} \int_{B_{R_n} \setminus B_R} |z_n^k|^p \leq \varepsilon \quad \text{and} \quad \int_{\mathbb{R}^N \setminus B_R} |v^k|^p \leq \varepsilon. \quad (2.10)$$

In view of (2.5), and taking R large enough, (2.10) also holds if we replace z_n^k by $(-a_n^k) \star u_{2,n}^k$.

Consider a smooth cut off function $\eta: [0, \infty) \rightarrow [0, 1]$ such that $\eta \equiv 1$ on $[0, 1]$ and $\eta \equiv 0$ on $[2, \infty)$. Set $v_n^k(x) := \eta(2|x|/R_n)v^k(x)$. Then

$$\lim_{n \rightarrow \infty} v_n^k = v^k \quad \text{in } H^1. \quad (2.11)$$

From the continuity of \mathcal{F} on $L^p(B_R)$, $v_n^k = v^k$ on B_R , and $\lim_{n \rightarrow \infty} z_n^k = v^k$ in $L^p(B_R)$ we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{B_R} |f(x, z_n^k) - f(x, z_n^k - v_n^k) - f(x, v_n^k)|^\nu dx \\ = \lim_{n \rightarrow \infty} \int_{B_R} |f(x, z_n^k) - f(x, z_n^k - v^k) - f(x, v^k)|^\nu dx = 0. \end{aligned}$$

Since $v_n^k \equiv 0$ in $\mathbb{R}^N \setminus B_{R_n}$, this in turn yields for any $\varepsilon > 0$ and R chosen accordingly, as in (2.10),

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |f(x, z_n^k) - f(x, z_n^k - v_n^k) - f(x, v_n^k)|^\nu dx \\ = \limsup_{n \rightarrow \infty} \int_{B_{R_n} \setminus B_R} |f(x, z_n^k) - f(x, z_n^k - v_n^k) - f(x, v_n^k)|^\nu dx \\ \leq C \limsup_{n \rightarrow \infty} \int_{B_{R_n} \setminus B_R} (|z_n^k|^\mu + |z_n^k - v_n^k|^\mu + |v_n^k|^\mu)^\nu \\ \leq C \limsup_{n \rightarrow \infty} \int_{B_{R_n} \setminus B_R} (|z_n^k|^p + |v^k|^p) \\ \leq C\varepsilon, \end{aligned}$$

where C is independent of ε . Letting ε tend to 0 and using (2.11) we obtain

$$\lim_{n \rightarrow \infty} |\mathcal{F}(z_n^k) - \mathcal{F}(z_n^k - v_n^k) - \mathcal{F}(v^k)|_\nu = 0.$$

Since (2.10) also holds for $(-a_n^k) \star u_{2,n}^k$ instead of z_n^k , the same arguments yield

$$\lim_{n \rightarrow \infty} |\mathcal{F}((-a_n^k) \star u_{2,n}^k) - \mathcal{F}((-a_n^k) \star u_{2,n}^k - v_n^k) - \mathcal{F}(v_n^k)|_\nu = 0.$$

Set $u_{i,n}^{k+1} := u_{i,n}^k - a_n^k \star v_n^k$. By the equivariance of \mathcal{F} and the invariance of the involved norms under the \mathbb{Z}^N -action,

$$\lim_{n \rightarrow \infty} |\mathcal{F}(u_{i,n}^k) - \mathcal{F}(u_{i,n}^{k+1}) - \mathcal{F}(a_n^k \star v_n^k)|_\nu = 0 \quad \text{for } i = 1, 2, \quad (2.12)$$

and, since $\|\cdot\|^2$ BL-splits,

$$\lim_{n \rightarrow \infty} |\|u_{i,n}^k\|^2 - \|u_{i,n}^{k+1}\|^2 - \|v_n^k\|^2| = 0 \quad \text{for } i = 1, 2. \quad (2.13)$$

Equations (2.13) and (2.4) (for k) imply that

$$\max_{i=1,2} \limsup_{n \rightarrow \infty} \|u_{i,n}^{k+1}\| \leq C_1,$$

hence (2.4) for $k+1$. The definition of the sequences $u_{i,n}^{k+1}$ and (2.5) (for k) imply that

$$\lim_{n \rightarrow \infty} |u_{1,n}^{k+1} - u_{2,n}^{k+1}|_p = \lim_{n \rightarrow \infty} |u_{1,n}^k - u_{2,n}^k|_p = 0,$$

hence (2.5) for $k+1$. It follows from (2.12) and (2.6) (for k) that

$$\liminf_{n \rightarrow \infty} |\mathcal{F}(u_{1,n}^{k+1}) - \mathcal{F}(u_{2,n}^{k+1})|_\nu = \liminf_{n \rightarrow \infty} |\mathcal{F}(u_{1,n}^k) - \mathcal{F}(u_{2,n}^k)|_\nu \geq C_2,$$

hence (2.6) for $k+1$. Last but not least, from (2.7) (for k), (2.8), and (2.9) it follows that

$$\lim_{n \rightarrow \infty} |a_n^\ell - a_n^k| = \infty \quad \text{if } \ell < k, \quad (2.14)$$

and hence by (2.7) (for k) and (2.11) that

$$\text{w-}\lim_{n \rightarrow \infty} (-a_n^\ell) \star u_{i,n}^{k+1} = \text{w-}\lim_{n \rightarrow \infty} ((-a_n^\ell) \star u_{i,n}^k - (a_n^k - a_n^\ell) \star v_n^k) = 0 \quad \text{in } H^1, \text{ if } \ell < k.$$

Moreover, by the definition of a_n^k ,

$$\text{w-}\lim_{n \rightarrow \infty} (-a_n^k) \star u_{i,n}^{k+1} = \text{w-}\lim_{n \rightarrow \infty} ((-a_n^k) \star u_{i,n}^k - v_n^k) = 0 \quad \text{in } H^1.$$

This proves (2.7) for $k+1$.

We now modify the sequences (a_n^k) , (v_n^k) , and $(u_{i,n}^{k+1})$ we have just built by taking away a finite number of terms at their start. By (2.5) and (2.6) we may arrange it so that

$$|u_{1,n}^{k+1} - u_{2,n}^{k+1}|_p \leq \frac{1}{k+1} \quad (2.15)$$

$$|\mathcal{F}(u_{1,n}^{k+1}) - \mathcal{F}(u_{2,n}^{k+1})|_\nu \geq C_2 - \frac{1}{k+1} \quad (2.16)$$

for all $n \in \mathbb{N}_0$. Since $\|\cdot\|^2$ BL-splits, (2.11) and (2.14) imply for any $\ell \leq k+1$ that

$$\lim_{n \rightarrow \infty} \left\| \sum_{j=\ell}^{k+1} a_n^j \star v_n^j \right\|^2 = \sum_{j=\ell}^{k+1} \|v^j\|^2.$$

We may therefore arrange it so that

$$\left\| \sum_{j=\ell}^{k+1} a_n^j \star v_n^j \right\|^2 \leq 2 \sum_{j=\ell}^{k+1} \|v^j\|^2, \quad \text{for all } n \text{ and } \ell \leq k+1. \quad (2.17)$$

Note that we need not and do not modify the sequences $(u_{i,n}^\ell)$, $\ell \leq k$, that were built in earlier steps.

Now we consider the process of constructing sequences as finished and proceed to prove properties of the whole set. Equation (2.13) leads to

$$\|u_{1,n}^{k+1}\|^2 = \|u_{1,n}^0\|^2 - \sum_{j=0}^k \|v^j\|^2 + o(1) \quad \text{as } n \rightarrow \infty,$$

and hence $\sum_{j=0}^\infty \|v^j\|^2 \leq C_1$ by (2.4). In view of (2.8) this yields

$$q_k \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (2.18)$$

We claim that the diagonal sequence $(u_{1,n}^n)$ satisfies

$$b_n \star u_{1,n}^n \rightarrow 0 \quad \text{in } H^1, \text{ as } n \rightarrow \infty, \text{ for every sequence } (b_n) \subseteq \mathbb{Z}. \quad (2.19)$$

Note that under our convention we have the representation

$$u_{1,n}^n = u_{1,n}^k - \sum_{j=k}^{n-1} a_n^j \star v_n^j \quad \text{if } n \geq k. \quad (2.20)$$

First we show that

$$\text{w-lim}_{n \rightarrow \infty} (-a_n^k) \star u_{1,n}^n = 0 \quad \text{in } H^1, \text{ for all } k \in \mathbb{N}_0. \quad (2.21)$$

Fix $k \in \mathbb{N}_0$. For every $w \in H^1$ and $\varepsilon > 0$ there is ℓ_0 such that

$$\|w\|^2 \sum_{j=\ell_0}^\infty \|v^j\|^2 \leq \varepsilon^2/2.$$

Then (2.17) and (2.20) yield for $n \geq \ell_0$

$$\begin{aligned}
& \left| \langle (-a_n^k) \star u_{1,n}^n, w \rangle \right| \\
& \leq \left| \langle (-a_n^k) \star u_{1,n}^{k+1}, w \rangle \right| + \left| \left\langle \sum_{j=k+1}^{\ell_0-1} (a_n^j - a_n^k) \star v_n^j, w \right\rangle \right| + \|w\| \left\| \sum_{j=\ell_0}^{n-1} a_n^j \star v_n^j \right\| \\
& \leq \left| \langle (-a_n^k) \star u_{1,n}^{k+1}, w \rangle \right| + \left| \left\langle \sum_{j=k+1}^{\ell_0-1} (a_n^j - a_n^k) \star v_n^j, w \right\rangle \right| + \varepsilon.
\end{aligned}$$

The first term in the last expression tends to 0 as $n \rightarrow \infty$ by (2.7), and the second term tends to 0 by (2.11) and (2.14). Since $\varepsilon > 0$ and $w \in H^1$ were arbitrary, this proves (2.21).

To finish the proof of (2.19), suppose for a contradiction that $w\text{-}\lim_{n \rightarrow \infty} b_n \star u_{1,n}^n = v \neq 0$ in H^1 , for a subsequence. Equation (2.21) implies that $\lim_{n \rightarrow \infty} |b_n + a_n^k| = \infty$, for every $k \in \mathbb{N}_0$. Pick $k \in \mathbb{N}_0$ such that $q_k < \|v\|$. This is possible by (2.18). Then, for every $w \in H^1$, it follows from (2.17) and (2.20) that

$$\left| \langle b_n \star u_{1,n}^k - v, w \rangle \right| \leq \left| \langle b_n \star u_{1,n}^n - v, w \rangle \right| + \left| \left\langle \sum_{j=k}^{n-1} (b_n + a_n^j) \star v_n^j, w \right\rangle \right| \rightarrow 0$$

as $n \rightarrow \infty$, similarly as above. Hence $w\text{-}\lim_{n \rightarrow \infty} b_n \star u_{1,n}^k = v$ with $\|v\| > q_k$, in contradiction with the definition of q_k . This proves (2.19).

We are now in the position to finish the proof of uniform continuity of \mathcal{F} . Equations (2.15) and (2.16) imply that

$$\begin{aligned}
& \lim_{n \rightarrow \infty} |u_{1,n}^n - u_{2,n}^n|_p = 0 \\
& \liminf_{n \rightarrow \infty} |\mathcal{F}(u_{1,n}^n) - \mathcal{F}(u_{2,n}^n)|_\nu \geq C_2.
\end{aligned}$$

By Lemma 2.1 and (2.19) $u_{1,n}^n \rightarrow 0$ in L^p . This contradicts the continuity of \mathcal{F} on L^p and therefore proves the assertion about uniform continuity.

It only remains to prove BL-splitting for \mathcal{F} . Suppose that $u_n \rightharpoonup v$ in H^1 . By the same arguments we used to obtain (2.12) there is a sequence $(v_n) \subseteq H^1$ such that $v_n \rightarrow v$ in H^1 and

$$\mathcal{F}(u_n) - \mathcal{F}(u_n - v_n) \rightarrow \mathcal{F}(v) \quad \text{in } L^\nu \tag{2.22}$$

as $n \rightarrow \infty$. Since (u_n) and (v_n) are bounded in H^1 , and by the uniform continuity of \mathcal{F} on bounded subsets of H^1 with respect to the L^p -norm (and hence also with respect to the H^1 -norm), it follows that we may replace v_n by v in (2.22). \square

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